Financial Transaction Tax: Policy Analytics based on Optimal Trading

Abstract

Introducing a financial transaction tax (FTT) has recently attracted tremendous attention. Both proponents and opponents dispute the dampening effects of a FTT on financial markets. In this paper, we proposed a model to show in some circumstances there exists a win-win situation via optimal trading when the tax burden can be dispersed. The way of absorbing FTT in our model is to adjust the bid-ask spread. In our optimal trading model considering FTT, the representative traders depend on liquidity (market depth) they supplied to weight their transaction cost associated by adjusting the spread ex post. We show the analytical properties and computational solutions of our model in finding the optimal trading strategy under different market situations to offset FTT. We conduct a simulation study to show the superior performance of our proposed optimal trading strategy comparing with the alternative strategies that do not consider absorbing FTT in their trading. The results show that there is a win-win situation and financial institutions will not be worse off if such an optimal trading strategy is applied to offset the FTT and reduce transaction cost.

Keywords: Bid-ask spread, Discrete optimization; Financial transaction tax (FTT); Optimal trading; Price impact

JEL classification: C61; C63; G10

1 Introduction

The financial sector has been considered as a major cause of the latest crisis. The European Commission on 28 September, 2012, put forward a proposal for a financial transaction tax (FTT)[1] attempting to make financial market players take more responsibility for resolving the crisis that they caused and to discourage excessive risk-taking in future. The European Parliament gave its consent to the proposal on 12 December, 2012.[2] Having obtained European Parliament's consent, the European commission will turn the FTT plans into reality after being authorized by the European Council.

Financial transaction taxes have been applied in many countries on spot share trading and derivatives transactions. For example, for equity, the stamp duty in UK, which is 0.5% tax on the value of spot transactions in shares of UK companies, while the tax rate (or fees) between 0.15% and 1% are applied in Australia, Brazil, China (0.1%), Hong Kong (0.1%), India (0.25%), Indonesia (0.1%), Ireland (1%), Switzerland (0.15%), Taiwan (0.3%), Singapore (0.2%), South Africa (0.25%), South Korea (0.5%), and US[3]. Particularly, the Taiwanese transaction tax is rather broad and covers various kinds of securities, including bonds and futures contracts. In addition, French amendment of FTT (from 0.1% to 0.2%) has been approved by the French National Assembly and the French Senate and been effective in 1 August, 2012.

The issue of whether a FTT should be levied has been controversial ever since it was suggested to reduce destabilization in equities by Keynes (1936) and destabilization in currency speculation by Tobin (1978). Schulmeister et al. (2008) point out that such a debate focuses on the answers to three questions. First, is there excessive trading in financial markets which drives prices to fluctuate excessively over the short run as well as over the long run? Second, could a small tax on financial transactions dampen destabilizing speculation without distorting liquidity beyond the level needed for market efficiency? Third, will the revenues of a general FTT even at a low tax rate be substantial relative to the costs of its implementation? There exists remarkable discrepancy between answers to each of these three questions based on theoretic and empirical investigation. Based on various perspectives about trading and price dynamics in asset markets and the effects of a transaction tax, FTT proponents state that a transaction tax increases the costs of speculative trades the more the higher their trading frequency, consequently, it could stabilize asset prices and thereby improve the overall macroeconomic performance. Relying on a perception of trading and price dynamics fundamentally different from that of the proponents, FTT opponents state that a transaction tax decreases liquidity in turn will increase the short-term volatility of asset prices, as a consequence, reduce market efficiency.

[1] A distinction could be made between a securities transaction tax (STT), a currency transaction tax (CTT), and a bank debit tax.

[2] The 11 participating countries are Austria, Belgium, Estonia, France, Germany, Greece, Italy, Portugal, Slovakia, Slovenia and Spain.

[3] International Bulletin for Fiscal Documentation, IMF, 2010.

In this paper, we build a model based on optimal trading to explain there does exit a win-win situation for policy maker and financial market players. The FTT is neither a panacea nor a damp squib. In our model, market liquidity could be preserved and liquidity supplier even can better off after conducting optimal trading strategy after introducing a transaction tax. In our model, oligopolistic financial institutions have obligation to continuously make market (i.e., to maintain liquidity). The objective is then to minimize transaction cost if financial institutions intend to maintain same liquidity level as before introducing FTT. Learning from market microstructure theory, market markers will enlarge their spread of quotations to compensate for the profit reduction caused by FTT. The spread is the difference between the best bid (highest price buyer is willing to pay) and best ask (lowest price a seller is willing to accept). Previous studies have shown that bid-ask spread is determined by trading activity, risk of holding inventory, adverse information, and competition. In order to maintain market liquidity, market makers (specialists) are compensated by buying at low price and selling at high price, that is, the spread must cover costs incurred by the market makers. In consequence, the bid-ask spread turns to be inferred directly from the transaction prices (see, for example, Schwartz 1988 and McInish and Wood (1992)). In the literature, the bid-ask spread is often assumed as a constant, see, for example, Bertsimas and Lo (1998), Almgren and Chriss (2000), and Obizhaeva and Wang (2012). Some empirical studies show that the bid-ask spread over a fixed time interval displays power-law distribution and long-range dependence, and there exists logarithmic relation between the bid-ask spread and the trade volume, see Plerou et al. (2005) and Ponzi et al. (2009). In our model, we assume market makers will adopt a linear spread function which contains two parts: a constant and a variation. The constant part will cover all common components (e.g., explicit costs and adverse selection) discovered in the literature. The variation part in our model depends on the previous trading volume. The more the market maker trade in previous trading period, the more the FTT compensation they require at current trading period.

Similar as Bertsimas and Lo (1998), Almgren and Chriss (1999, 2000), Obizhaeva and Wang (2012), and Sun et al. (2012), we assume the market player faces a linear price impact function, that is, a linear combination of the permanent and temporary price impact shall be characterized by a price impact function. Hubermann and Stanzl (2004) indicate that the linear price impact function excludes the quasi-arbitrage (i.e., price manipulation) and supports viable market prices. We also follow other studies (see, for example, Huberman and Stanzl (2005) and Obizhaeva and Wang (2012)) focusing on the discrete-time setting of model which will help us to derive the computational solution. In our model, we intend not to explain the price volatility caused by a transaction tax due to the discrepant perception of FTT on price hold by proponents and opponents. We consider the price is exogenous and decided by the market. We assume two different scenarios for the underlying price dynamics, that is, the underlying price change follows the Brownian motion as suggested by Obizhaeva and Wang (2012) and geometric Brownian motion as suggested by Ting et al. (2007) and Sun et al. (2012).

In this paper, we consider three different trading strategies for the financial institutions that make the market by supplying liquidity in the oligopolistic market. The first trading strategy is the naive trading, which simply splits large orders to equalized small pieces to reduce price impact without considering any optimal adjustment. The second trading strategy is based on the model proposed by Obizhaeva and Wang (2012), which keeps a constant spread assumption and adopts optimal trading adjustment to reduce transaction cost. The third trading strategy is our proposed optimal trading strategy, which considers the influence of FTT on spread (i.e., described by a linear spread function) and optimally chooses trading strategy to reduce the transaction cost. We show the importance of incorporating the linear spread function in finding the optimal solution under these two price dynamics by displaying the superior performance of our strategy over two other strategies. The volume weighted average price (VWAP) is used as the benchmark to decide the during-trading cost measure of these trading strategies (see, for example Werner (2003) and Goldstein et al. (2009)). Only the trading strategy with lowest (highest) VWAP for buying (selling) is preferred. The numerical example we conducted indicate that based on the VWAP benchmark the overall performance of our optimal trading strategy dominates the alternative trading strategies investigated in this study. It implies that market players can improve their profitability by applying optimal trading strategy after introducing a financial transaction tax without distorting market liquidity.

We organize the paper as follows: The fundamental model setup is provided in Section 2. In Section 3, we describe our contributions and introduce the analytical solutions for solving the optimization problem of minimizing the transaction cost, that is, we consider the linear bid-ask spread function and allow the underlying price dynamics follows both Brownian motion and geometric Brownian motion. In Section 4, we investigate the performance of our model by running large sample simulations and report our numerical results. We summarize our conclusions in Section 5.

2 The Model Setup

Algorithmic trading (AT) utilizes computers to transact within a finer time interval at incomprehensible speed. AT firms represent approximately 2% of the nearly 20,000 trading firms operating in the U.S. markets, but since 2009 they have accounted for over 70% of the volume in U.S. equity markets and are fast approaching 50% of the volume in futures markets[4]. It could be an evidence for proponents of FTT to show there exists excessive volatility (see, for example, Schulmeister (2009)), particulary after the flash crash in May 6, 2010. As algorithmic trading plays an important role in daily trading business, in our model, we assume there is a representative trader who can apply algorithms to find the optimal trading strategy. In our model, the representative trader needs to provide liquidity to accumulate (or liquidate) a large position of securities with size $X_0 (X_0 > 0)$ during a given time interval $[0, T] (T > 0)$. T refers to any trading period, for example, it could be one second or one day.

[4]CFTC. 2010. Proposed Rules, Federal Register, 75 (112) (June 11): 33198-33202.

The debate over the effect of FTT lasts long while. Proponents assert that (1) FTT will reduce excessive trading activities and short-term speculation as it could increase the costs of speculative trades; (2) FTT will stabilize asset prices and thereby improve the overall macroeconomic performance; and (3) FTT will provide considerable tax revenues which can be used for fiscal consolidation. On the other hand, opponents assert that (1) FTT will increase transaction costs and cause liquidity to decline which in turn will increase the short-term volatility of asset prices; and (2) FTT is hard to implement, particularly for international transactions. How does this sweeping market change of introducing a FTT affect financial institutions? There could be two very different ways. Supporters claim that transaction cost will be reduced due to price stabilization. On the other side, detractors claim that transaction cost will increase due to the liquidity reduction after introducing FTT. To find out, we build a model focusing on the transaction cost to investigate the performance of financial institution after introducing a FTT. In order to make our model remains neutral (i.e., toward neither supporter nor detractor's viewpoint to build the model), we have several assumptions that take into account both supporter and detractor's consideration for FTT.

2.1 Assumptions

One of the issues suggested by Schulmeister et al. (2008) considering FTT to be checked is the existence of excessive trading in financial market which causes excessive price fluctuation. Schulmeister (2009) based on evaluating of the empirical evidence concludes that asset markets show excessive liquidity and excessive price volatility. However, there is no unified way to measure the liquidity and volatility rather the excessive liquidity and volatility. From the market microstructure literature, see, for example, Harris (2003) immediacy, depth, width, and resiliency are liquidity dimensions and volatility can be distinguished between fundamental volatility and transitory volatility. Without carefully defining the liquidity and volatility, the dispute on FTT turns to blur the real effects of FTT. In order to avoid the ambiguity when defining liquidity and volatility and make our model more practical, we consider both liquidity (in our model we focus on the market depth q) and volatility are exogenous variables.

In our model, we assume a representative trader, who could be a financial institution has proprietary and flow trading desks or a specialist to supply liquidity to the market by continuously offering quotes. We assume it is an oligopolistic market. Our representative trader itself cannot decide the price but his trading does have price impact. In fact, the representative trader can be treated essentially as a dealer who supplies liquidity in an oligopolistic market. In our model, as we are going to investigate FTT neutrally without considering its suspicion of liquidity reduction, we allow our representative trader trade only with market orders. Market order will be executed at the best price currently available in the market and it is used to realize immediacy of a trade. The advantage of this setting for our model is twofold. First, it reduce the difficulty of the algorithm run by our trader to estimate the execution uncertainty of limit orders, particularly when the market bid-ask spread is relatively large. Second, it can limit the trader from flickering quotes and make them bring liquidity to the market.

In our model, the trader is only allowed to submit market orders at discrete time points which are equidistantly distributed, this means the trader trades at time point $N+1$ during the whole trading period. $N \in \{1, 2, \ldots\}$ stands for the trading frequency. $t_i (i \in 0, \ldots, N)$ are time points starting at $t_0 = 0$ and ending in $t_N = T$. We then write $t_i = i\tau$, where $\tau = T/N$ is the duration between two successive time points we are able to trade. We define x_{t_n} as the size of the order the trader has submitted at time point t_n, then $X_0 = \sum_{n=0}^{N} x_{t_n}$. We define $X_{t_n} = X_0 - \sum_{i=0}^{n-1} x_{t_i}$ as the position the trader still needs to accumulate before t_n. We also assume $x_{t_n} \geq 0$. The space of feasible strategies is then defined as follows:

$$\Phi = \left\{ \{x_{t_0}, \ldots, x_{t_N}\} : x_{t_n} \geq 0 \; \forall \; n \in \{0, \ldots, N\} \, ; \sum_{n=0}^{N} x_{t_n} = X_0 \right\}.$$

In this paper we consider the underlying price dynamics follow (1) a Brownian motion F_t and (2) a geometric Brownian motion F_t with a given drift μ, variance σ, and initial value $F_0 = V_0$. In addition, part of the price movement is the distance of the spread $S(\cdot) := \alpha x_{n-1} + \beta$.

2.2 Spread function $S(\cdot)$

Obizhaeva and Wang (2012) describe the supply/demand dynamics shown by limit order book. Let A be the best ask price and B the best bid, we can define the mid-quote price V as $V = (A+B)/2$ and spread s as $s = A - B$. Then $A = V + s/2$ and $B = V - s/2$. In this paper, we consider the representative trader's execution of buy order, we then focus on the ask price A. In the literature, the bid-ask spread is often assumed to be a constant, see, for example, Bertsimas and Lo (1998), Almgren and Chriss (2000), and Obizhaeva and Wang (2012). Some empirical studies show that the bid-ask spread over a fixed time interval displays power-law distribution and long-range dependence, and there exists logarithmic relation between the bid-ask spread and the trade volume, see Plerou et al. (2005) and Ponzi et al. (2009). Based on these empirical findings, in this paper we consider the bid-ask spread not to be a constant but a linear function of previous trading volume. Using a linear function allows us to describe the movement of bid-ask spread by decomposing the spread components into tax based variance and trade variance. In this paper, we describe the bid-ask spread as a linear function $S(\cdot) := \alpha x_{n-1} + \beta$. The term αx_{n-1} stands for the tax based variance of the spread. Term α could (but not only) be the tax rate and x_{n-1} is the volume of the trader's previous trade. Term β is the trade variance of the spread and previous studies have shown that it is determined by trading activity, risk of holding inventory, adverse information, and competition. In order to maintain market liquidity, specialists or market makers are compensated by buying at low price and selling at high price, that is, the trade variance must cover costs incurred by the trader. The bid-ask spread turns to be inferred directly from the transaction prices (see, for example, Schwartz 1988 and McInish and Wood (1992)) and our setting of the bid-ask spread also considers the influence of FTT on transaction prices.

As we have pointed out, the representative trader in our model can be treated essentially as a dealer who supplies liquidity in an oligopolistic market. The spread then ultimately depends on the costs of running his business as dealers set their spreads to maximize their profits and the spreads must wide enough to allow them to recover their costs of running business. We can decompose the dealer's bid-ask spread into two components: one for transaction cost and the other for the adverse selection that incurs when the trader trades with better-informed traders. The tax burden of FTT will increase the transaction cost of the dealer. The proponents of FTT also assert a uniform tax per transaction increases the costs of speculative trades. The higher the speculative trading frequency is, the more the increase in cost. We could say the FTT reinforce the adverse selection effect. Assuming FTT can reduce speculation, the investors who trade with the dealer under FTT must believe they know fundamental values well. If the dealer recognizes this asymmetrical information, he will widen the spread to recoup himself for supplying liquidity to better-informed traders at less attractive prices. Given there are three time points $t-1$, t, and $t+1$, at time $t-1$ the dealer estimates the asset value, bid, and ask using information currently available then submit the bid/ask quotes. At time t the dealer can confirm the accuracy of his estimation made at time $t-1$ and know if he has traded with a better-informed trader and the relevant cost/lose. Then at time $t+1$, he will quotes new bid/ask which contains not only the information currently available at time $t+1$ but also the compensation for the previous cost/lose at time t. The more the trader transacts at time t, the more the compensation he asks for recovering the cost/loss at time $t+1$. Our linear spread function $S(\cdot) := \alpha x_{n-1} + \beta$ can capture this argument, where α stands for the augmented FTT effect which is larger than the tax rate and its influence on the current bid/ask spread depends on the previous trading volume and β stands for other explicit costs. The best bid A_t, can be expressed by $A_t = F_t + (\alpha x_{n-1} + \beta)/2$.

2.3 Price impact

Similar to Sun et al. (2012), the limit order book is modeled by using a constant market depth q, which means when executing a buy order of the size q the price will be increased by 1 unit. In general, this translates into the price impact of an order x_{t_n} is x_{t_n}/q. The average price impact of the whole order is then $x_{t_n}/2q$. The price jump is due to the influence of order submitted to the market and consists of the permanent and temporary price impact.

We decompose the price impact of an order x_{t_n} into two parts $x_{t_n}/q = \lambda x_{t_n} + \kappa x_{t_n}$, where $0 \leq \lambda \leq 1/q$ is the percentage of the permanent price impact and $\kappa = 1/q - \lambda$ is the percentage of the temporary price impact contributed respectively to the total price impact. We call the term λx_{t_n} with $0 \leq \lambda \leq 1/q$ the permanent price impact of the trade x_{t_n}, and κx_{t_n} with $\kappa = 1/q - \lambda$, the temporary price impact. Permanent price impact is the change in price caused by the trader's order that leads the market to believe that future prices will be different than originally expected or there is a change in the asset's intrinsic value. Temporary price impact occurs whenever an order is released to the market but does not provide fundamental news or information that changes the market current valuation or long-run outlook of the underlying asset. Trades cause temporary increases in price for buy orders and temporary decrease for sell orders subsequently followed by a price reversion back to the initial price trajectory. In order to model the way the temporary price impact of an order vanishes along with time we use a resilience factor ρ following Obizhaeva and Wang (2012). The part of the temporary price impact of the order x_{t_n} that remains until $t > t_n$ is $\kappa x_{t_n} e^{-\rho(t-t_n)}$, where the resilience factor $\rho > 0$. The temporary price impact at time point t_n before we make a trade is defined as $D_{t_n} = \sum_{i=0}^{n-1} x_{t_i} \kappa e^{-\rho(t_n - t_i)}$. In fact, the temporary price impact D_{t_n} at t_n before we submit the order, satisfies the recursive equation $D_{t_n} = (D_{t_{n-1}} + \kappa x_{t_{n-1}})e^{-\rho\tau}$ with the initial condition $D_0 = 0$.

2.4 Execution cost and the objective function

In this model, we follow the finite resilience effect, which can be captured by the refresh rate ρ, of the limit order book. Obizhaeva and Wang (2012) point out that although the dynamics of the limit order book (i.e., the price impact of order) is given without additional equilibrium justification, the finite resilience is consistent with the behavior obtained from simple equilibrium models. Once the trade size and frequency are predefined for the admissible trading strategies in these existing equilibrium models, there is no adequate way to provide the optimal execution solution. In this paper, we focus on the knowledge of optimal trading under general limit order book dynamics, that is, the dynamics driven by price impact. Our model setting is flexible and allows speculative trading and different dynamics for its time evolution in response to the speculative trading. Since our main goal of this paper is to demonstrate the influence of FTT in determining the optimal trading of trader, we only focus on a specific situation from the general setting. Alfonsi et al. (2010) provide a more general model setting with respect to the limit order book dynamics.

Let $\bar{P}_n$ be the average execution price for x_{t_n}, the goal of our representative trader is to minimize the expected total cost of his purchase by choosing an optimal strategy over a given trading horizon T. That is,

$$\min_{x_0,\ldots,x_T} E\Big(\sum_{n=0}^{N} \bar{P}_n x_n\Big),$$

which implies that the risk-neutral traders only consider the expected value rather than the uncertainty of the total transaction cost. Given the order book dynamics (i.e., price impact of order) we described above, we can define the total transaction cost of a trading strategy for a given order size X_0. We can see that the cost for a single trade x_{t_n} can be expressed as

$$c(x_{t_n}) = \int_0^{x_{t_n}} P_{t_n}(x)dx,$$

where $P_{t_n}(x)$ is given by

$$x = \int_{A_t}^{P_{t_n}(x)} q_t(P)dP.$$

We have $P_t(x) = A_t + x/q$ and

$$c(x_{t_n}) = \big(A_{t_n} + x_{t_n}/(2q)\big)x_{t_n}.$$

The total cost of $N+1$ trades of size x_{t_n}; $n = 0, 1, \ldots, N$, is $\sum_{n=0}^{N} c(x_{t_n})$. We can obtain the following objective function the trader has is to minimize the expected cost of the whole order under FTT, that is,

$$\min_{x_0, \ldots, x_T} E\Big(\sum_{n=0}^{N} x_{t_n}\big(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2} + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{x_{t_n}}{2q})\Big).$$

3 Analytical solutions

In our model, the dynamics of bid-ask spread is described by a linear function of the previous trade $S(\cdot) = \alpha x_{n-1} + \beta$, since the representative trader is the large trader and could influence the limit order book. In this section, we derive our analytical solutions based on different assumptions, that are, (1) when the dynamics of underlying price follows Brownian motion; (2) when the dynamics of underlying price follows geometric Brownian motion; and (3) when considering "no short-selling" constraint.

3.1 Case for Brownian motion

When the dynamics of underlying price change follows a Brownian motion, we obtain the following proposition.

Proposition 1 *Given the model setting described in Section 2 with the underlying value following a Brownian motion, the strategy $x_{t_N} = X_{t_N}$, and*

$$\begin{aligned} x_{t_n} &= -\frac{1}{2}\delta_{n+1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})X_{t_n} \\ &\quad -\frac{1}{2}\delta_{n+1}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau})D_{t_n} - \frac{1}{2}\delta_{n+1}(\frac{\alpha x_{t_{n-1}}}{2}), \end{aligned} \tag{1}$$

with $x_{-1} = 0$, is optimal, if $(x_{t_0}, \ldots, x_{t_N}) \in \Phi$. Then the optimal value function has the form

$$\begin{aligned} &J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) \\ = \;& (F_n + \frac{\beta}{2})X_{t_n} + \lambda X_0 X_{t_n} + d_n X_{t_n}^2 + l_n X_{t_n} D_{t_n} + f_n D_{t_n}^2 + h_n X_{t_n} x_{t_{n-1}} + s_n D_{t_n} x_{t_{n-1}} + u_n x_{t_{n-1}}^2, \end{aligned} \tag{2}$$

with coefficients defined by

$$
\begin{aligned}
d_n &= d_{n+1} - \frac{1}{4}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})^2, \\
l_n &= l_{n+1}e^{-\rho\tau} - \frac{1}{2}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}), \\
f_n &= f_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}^{-1}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau})^2, \\
h_n &= -\frac{\alpha}{4}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1}), \\
s_n &= -\frac{\alpha}{4}\delta_{n+1}^{-1}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}), \\
u_n &= -\frac{\alpha^2}{16}\delta_{n+1}^{-1}, \\
\delta_{n+1} &= (\frac{1}{2q} + d_{n+1} - \kappa l_{n+1}e^{-\rho\tau} + \kappa^2 f_{n+1}e^{-2\rho\tau} - h_{n+1} + \kappa s_{n+1}e^{-\rho\tau} + u_{n+1})^{-1}, \qquad (3)
\end{aligned}
$$

and terminal conditions

$$
d_N = (\frac{1}{2q} - \lambda), l_N = 1, f_N = 0, h_N = \frac{\alpha}{2}, s_N = 0, \text{ and } u_N = 0. \qquad (4)
$$

Proof 1 *See Appendix A.*

3.2 Case for geometric Brownian motion

In this section we allow the underlying price movement to follow a geometric Brownian motion, then we obtain the following proposition.

Proposition 2 *Given the model setting described in Section 2 with the underlying value following a geometric Brownian motion, the strategy $x_{t_N} = X_{t_N}$, and*

$$
\begin{aligned}
x_{t_n} &= -\frac{1}{2}\delta_{n+1}[(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau})D_{t_n} + (-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa + v_{n+1})X_{t_n} \\
&\quad +(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a)F_{t_n} + (\frac{\alpha x_{t_{n-1}}}{2})] \qquad (5)
\end{aligned}
$$

with $x_{-1} = 0$, is optimal, if $(x_{t_0}, \ldots, x_{t_N}) \in \Phi$. Then the optimal value function has the form

$$
\begin{aligned}
& J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) \\
= \;& (a^{N-n}F_n + \frac{\beta}{2})X_{t_n} + \lambda X_0 X_{t_n} + b_n X_{t_n}^2 + c_n D_{t_n}^2 + d_n F_{t_n}^2 + g_n X_{t_n} D_{t_n} \\
& + h_n X_{t_n} F_{t_n} + l_n D_{t_n} F_{t_n} + v_n X_{t_n} x_{t_{n-1}} + w_n F_{t_n} x_{t_{n-1}} + s_n D_{t_n} x_{t_{n-1}} + u_n x_{t_{n-1}}^2, \qquad (6)
\end{aligned}
$$

with $a = e^{\mu\tau}$, $m = e^{(2\mu+\sigma^2)\times\tau}$, *and the coefficients are given as follows:*

$$
\begin{aligned}
b_n &= b_{n+1} - \frac{1}{4}\delta_{n+1}(-\lambda - 2b_{n+1} + g_{n+1}\kappa e^{-\rho\tau} + v_{n+1})^2, \\
c_n &= c_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}(1 + 2\kappa c_{n+1}e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau})^2, \\
d_n &= d_{n+1}m - \frac{1}{4}\delta_{n+1}(1 - a^{N-n} - h_{n+1}a + l_{n+1}e^{-\rho\tau}a\kappa + w_{n+1}a)^2, \\
g_n &= g_{n+1}e^{-\rho\tau} \\
&\quad -\frac{1}{2}\delta_{n+1}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa + v_{n+1})(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}), \\
h_n &= h_{n+1}a \\
&\quad -\frac{1}{2}\delta_{n+1}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa + v_{n+1})(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a) \\
l_n &= l_{n+1}e^{-\rho\tau}a \\
&\quad -\frac{1}{2}\delta_{n+1}(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a)(1 + 2c_{n+1}\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}), \\
v_n &= -\frac{1}{2}\delta_{n+1}\frac{\alpha}{2}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa + v_{n+1}), \\
s_n &= -\frac{1}{2}\delta_{n+1}\frac{\alpha}{2}(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}), \\
w_n &= -\frac{1}{2}\delta_{n+1}\frac{\alpha}{2}(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a), \\
u_n &= -\frac{1}{4}\delta_{n+1}(\frac{\alpha^2}{4}), \\
\delta_{n+1} &= (\frac{1}{2q} + b_{n+1} - g_{n+1}\kappa e^{-\rho\tau} + c_{n+1}\kappa^2 e^{-2\rho\tau} - v_{n+1} + s_{n+1}\kappa e^{-\rho\tau} + u_{n+1})^{-1},
\end{aligned} \tag{7}
$$

with initial conditions

$$
b_N = \frac{1}{2q} - \lambda,\ c_N = 0,\ d_N = 0,\ g_N = 1,\ h_N = 0,\ l_N = 0,\ v_N = \frac{\alpha}{2},\ s_N = 0,\ w_N = 0,\ \text{and } u_N = 0. \tag{8}
$$

Proof 2 *See Appendix B.*

3.3 Case for "no short-selling" constraint

In order to make our model more realistic, we allow the "no short selling" constraint. "No short selling" refers to the situation that before the purchase position is fully completed, the trader is not allowed to sell. Similarly, we can use this constraint for not purchasing before the sell position is fully completed. It is a practical issue, particularly when a trader conducts flow trading for his clients. In this model we face the same situation as stated by Sun et al. (2012), then we obtain following propositions when the dynamics of underlying price change follows Brownian motion.

There are three different cases for the size of a trade x_n in order to realize the "no short selling" constraint. In the case that $x_n < 0$, we simply use $x_n = 0$. This value is the best solution for our problem, because the optimal function at t_n is parabola opened upward based on the control variable x_n. For the case that $x_n < 0$, where x_n is the angular point, the value function J_{t_n} is increasing with larger values of x_n in the interval $[0, X_{t_n}]$. This infers the best possible solution is $x_n = 0$. For the case $0 < x_n < X_{t_n}$, the solution we have derived is still optimal. For the case $X_{t_n} < x_n$, the value function of J_{t_n} is decreasing with larger values of x_n in the interval $[0, X_{t_n}]$, the best possible solution is to set $x_n = X_{t_n}$. The coefficients for the modified strategy with the "no selling" constraint are given by following propositions. Using these coefficients we can calculate all possible strategies, and then select the feasible strategy with the lowest expected cost.

Proposition 3 *For $x_{t_n} = 0$ we have following coefficients for the optimal value function in Proposition 1*

$$d_n = d_{n+1},\ l_n = l_{n+1}e^{-\rho\tau},\ f_n = f_{n+1}e^{-2\rho\tau},\ h_n = 0,\ s_n = 0,\ u_n = 0.$$

Proof 3 *Entering $x_{t_n} = 0$ into Equation 9 then we obtain*

$$\begin{aligned} J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) &= (F_n + \frac{\beta}{2})X_{t_n} + \lambda X_0 X_{t_n} + d_{n+1}X_{t_n}^2 \\ &\quad + l_{n+1}X_{t_n}D_{t_n}e^{-\rho\tau} + f_{n+1}D_{t_n}^2 e^{-2\rho\tau} \end{aligned}$$

From this we conclude the proof.

For the case $x_{t_n} = X_{t_n}$, we have following proposition.

Proposition 4 *For $x_{t_n} = X_{t_n}$ we have the coefficients*

$$d_n = \frac{1}{2q} - \lambda,\ l_n = 1,\ l_n = 0,\ h_n = \frac{\alpha}{2},\ s_n = 0, and\ u_n = 0.$$

Proof 4 *For $x_{t_n} = X_{t_n}$ the optimal value function takes the form*

$$J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) = [(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2}) + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{X_{t_n}}{2q}]X_{t_n},$$

leads to the given coefficients.

When the dynamics of underlying price change follows geometric Brownian motion, we obtain following propositions.

Proposition 5 *For $x_{t_n} = 0$ we have the coefficients for the optimal value function in Proposition 2*

$$b_n = b_{n+1},\ c_n = c_{n+1}e^{-2\rho\tau},\ d_n = d_{n+1}m,\ g_n = g_{n+1}e^{-\rho\tau},\ h_n = h_{n+1}a,\ l_n = l_{n+1}e^{-\rho\tau}a,\ v_n = 0,$$

$$s_n = 0,\ w_n = 0,\ u_n = 0.$$

Proof 5 *Entering $x_{t_n} = 0$ into Equation 12 then we obtain the optimal value function*

$$\begin{aligned} J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) &= (a \times a^{N-(n+1)} F_n + \frac{\beta}{2})(X_{t_n}) + \lambda X_0(X_{t_n}) + b_{n+1}(X_{t_n})^2 \\ &\quad + c_{n+1}(D_{t_n})^2 e^{-2\rho\tau} + d_{n+1} m F_{t_n}^2 + g_{n+1}(X_{t_n})(D_{t_n}) e^{-\rho\tau} \\ &\quad + h_{n+1}(X_{t_n}) a F_{t_n} + l_{n+1}(D_{t_n}) e^{-\rho\tau} a F_{t_n} \\ &= (a^{N-n} F_n + \frac{\beta}{2}) X_{t_n} + \lambda X_0 X_{t_n} + b_{n+1} X_{t_n}^2 \\ &\quad + c_{n+1} D_{t_n}^2 e^{-2\rho\tau} + d_{n+1} m F_{t_n}^2 + g_{n+1} X_{t_n} D_{t_n} e^{-\rho\tau} \\ &\quad + h_{n+1} X_{t_n} a F_{t_n} + l_{n+1} D_{t_n} e^{-\rho\tau} a F_{t_n}. \end{aligned}$$

From this we obtain the proposition.

For the case $x_{t_n} = X_{t_n}$, we have the following proposition.

Proposition 6 *For $x_{t_n} = X_{t_n}$ we have the coefficients*

$$b_n = \frac{1}{2q} - \lambda,\ c_n = 0,\ d_n = 0,\ g_n = 1,\ h_n = 0,\ l_n = 0,\ v_n = \frac{\alpha}{2}$$

$$s_n = 0,\ w_n = 0, and\ u_n = 0.$$

Proof 6 *For $x_{t_n} = X_{t_n}$ the optimal value function takes the form*

$$J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) = [(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2}) + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{X_{t_n}}{2q}] X_{t_n},$$

where we can obtain the coefficients as given by the proposition.

4 Simulation Study

We conduct an empirical study based on simulations in order to investigate the performance of our trading strategy under FTT. We use same parameters in Obizhaeva and Wang (2012), that are, $q = 5,000$, $\lambda = 1/2q$, $\kappa = 1/q - \lambda$, $\rho = 2.2$, $N = 10$, $X_0 = 100,000$, and $T = 1$. In our model, we assume two types of underlying price movement, i.e., Brownian motion and geometric Brownian motion. When the underlying price movement follows the geometric Brownian motion, we use variance $\sigma = 0.03\%$ and drift $\mu = 0.03\%$. Order behavior of our optimal trading strategy (S-K-Y in short) for different values of α and β, which describe the bid-ask spread changes, are shown in Table 1 and 2 respectively. Figure 1 and 2 illustrate the order behavior under Brownian motion and geometric Brownian motion with differnt α values. Order behaviors of Obizhaeva and Wang (2012) (O-W in short) and naive trading strategy (equally splitting the orders) are shown in Table 3. We see that when α is deminishing, our order behavior turns to be more closer to the U-shape behavior as that of Obizhaeva and Wang (2012). When α equals to zero, our order behavior is coincided with that of Obizhaeva and Wang (2012) when the underlying price movement follows Brownian motion. When the bid-ask spread changes significantly, e.g.,

α increases, the order behavior of our optimal order submission strategy turns not to be the U-shape and depends on the change of bid-ask spread.

From Table 2 we can see that the parameter β in bid-ask spread function $S(\cdot)$ has no impact on the trading strategy. As we have shown in our analytical solution, β does not occur in the optimal solution. This can be explained as our optimal order submission strategy is not influenced by the explicit transaction costs (e.g., commission or fee). It coincides with the observations from trading practice.

We compare the performance of our strategy with two alternative strategies (i.e., O-W and Naive). We use the volume weighted average price (VWAP) as the benchmark to decide the during-trading cost measure of these trading strategies (see, for example Werner (2003) and Goldstein et al. (2009)). Only the trading strategy with lowest VWAP (for buying in our simulation) is preferred. We use the parameters setting mentioned above and run the simulation for 10,000 and 100,000 times. We report our results in Table 4 and 5. We see that our strategy is significantly superior to both alternative strategies when the underlying price movement following the Brownian motion. We illustrate this conclusion in Figure 3. As we have mention, when α equals to zero, our trading strategy coincides with the strategy given by Obizhaeva and Wang (2012) when the underlying price movement following Brownian motion. When the underlying price movement following geometric Brownian motion, our strategy is also significantly superior to the alternatives. Figure 4 illustrate this conclusion when the drift (i.e., μ) in the geometric Brownian motion is negative and Figure 5 illustrate when there is a positive drift in the geometric Brownian motion.

So far we show that the representative trader (i.e., the dealer-type specialist or market maker) can reduce their transaction cost by applying optimal trading strategy after introducing FTT. It is interesting to learn the relationship between transaction cost and market liquidity under FTT. Market liquidity has several dimensions and captures the aspects of immediacy, breath, depth, and resiliency in markets. Immediacy refers to the speed with which a trade of a given size and cost can be completed. Breadth refers to the costs of doing a trade of a given size. Depth refers to the maximum size of a trade for any given bid/ask spread. Resiliency refers to how quickly prices revert to its former level after a large transaction. In this study, we only consider resiliency ρ. We run a simulation and show the results in Figure 6. Figure 6 contains two panels, the left panel illustrates the case when the underlying price dynamics is a Brownian motion and right panel for the geometric Brownian motion ($\mu = 3\%$).

For both panels of Figure 6, the relationship between transaction cost and resiliency can be expressed by a convex curve. It states that the transaction cost will be reduced when the resiliency increases, that is, the higher the market liquidity the lower the transaction cost. This confirms that our model does not violate the basic observation of market microstructure. When introducing FTT, i.e., α turns to be larger in our spread function $S(\cdot)$, the transaction cost will increase. From Figure 6, we can see that a smaller α requires less resiliency to maintain the same level of transaction cost. For example, the change of resiliency is much smaller when α increases from 0.01 bp to 0.05 bp than the change of resiliency when α increases from 0.01 bp to 1 bp. It implies that a large FTT will squeeze liquidity comparing with a small FTT. As discussed in literature, FTT can raise tax revenue and reduce the (speculative) trading activity. Darvas and von Weizsäcker (2011) also pointed out that a smaller FTT is preferred. Our results are consistent with their conclusion.

5 Conclusions

In this paper, we proposed a model to show in some circumstances there do exists a win-win situation via optimal trading when the financial transaction tax (FTT) burden can be absorbed by a dealer-type market maker in an oligopolistic market. The way of absorbing FTT in our model is to adjust the dealer's bid-ask spread which considers the effect caused by FTT. In our optimal trading model for FTT, the dealer-type market maker depend on liquidity (market depth) they supplied to minimize their transaction cost associated by adjusting the spread ex post.

We show the analytical properties and computational solutions of our model in finding the optimal trading strategy under different market situations to offset FTT. We conduct a simulation study to show the superior performance of our proposed optimal trading strategy comparing with the alternative strategies that do not consider absorbing FTT in their trading. The results show that there is a win-win situation: the tax revenue can be collected and the dealer-type market maker will not be worse off if such an optimal trading strategy is applied to offset the FTT and to reduce transaction cost. In addition, our results are compatible with the literature about FTT.

In this paper, we focus on the optimal execution problem under FTT faced by an representative trader who supplies liquidity (i.e., execute a large order) over a given period of time. Considering the finite resilience effect (i.e., price impact of the order), we model the reaction to the FTT with a linear spread function which depends on the dealer's total transaction cost (i.e., explicit cost, tax burden, and adverse selection component). The dynamics of spread function and resilience effect determine the optimal trading strategy. We consider various aspects of liquidity which might influence trading strategy based on the consequences caused by FTT. We demonstrate that for optimally chosen trading times, the optimal trading strategy under FTT can be determined by the resilience. The discrete orders submitted by the representative trader can be adjusted to capture the market liquidity changes induced by FTT.

Since FTT has been generally considered to be used for reducing speculative trading activity in financial markets, our model explicitly incorporates three dimensions of liquidity documented in market microstructure literature: market depth, bid-ask spread, and resilience to investigate the influence of FTT. The first dimension, i.e., market depth, captures the static aspect of liquidity. The last dimension, resilience, reflects the dynamic aspect of liquidity. We introduce a linear bid-ask spread function in this paper, which captures both static (by the constant term) and dynamic aspects (by the variable term) of liquidity in response to FTT. In our model, the variable part of the bid-ask spread determines how much the previous trade will cause current price changes when considering FTT. Market depth and the constant part of our bid-ask spread determine how much the current price changes in response to a trade and they are the key factors for the representative trader to decide the transaction cost. The resilience illustrates how future price evolves in response to the current trade. In our model, the price impact is assumed to dissipate gradually over time. We show that the optimal trading strategy depending crucially on the dynamic properties of the limit order book. The impact of FTT on transaction cost can be eliminated, particularly, when the tax rate is small. In other words, FTT can only damper the uninformed speculation, i.e., noise trading, and its influence on informed speculation can be marginalized. Since our model setting does not consider the equilibrium of the market, we could not say anything about whether FTT would reduce the efficiency of market price discovery process. However, we do show that there exits a win-win situation when introducing FTT, that is, the tax burden can be dispersed by applying optimal trading strategy and the tax revenue can be collected.

References

[1] Alfonsi, A., Fruth, A., Schied, A. (2010). Optimal execution strategies in limit order books with general shape functions. Quantitative Finance, 10(2), 143-157.

[2] Almgren, R., Chriss, N. (1999). Value under liquidation. Risk, 12, 61-63.

[3] Almgren, R., Chriss, N. (2000). Optimal execution of portfolios. Journal of Risk, 3(2), 5-39.

[4] Almgren, R. (2003). Optimal execution with nonlinear impact functions and trading enhanced risk. Applied Mathematical Finance, 10, 1-18.

[5] Almgren, R., Lorenz, J. (2007). Adaptive arrival price. In B. Bruce (Ed). Algorithmic Trading III: Precision, Control, Execution. New York: Institutional Investor Journals.

[6] Berkowitz, S. A., Logue, D. E., Noser, E. A. (1988). The total costs of transacting on the NYSE. Journal of Finance, 43, 97-112.

[7] Bertsimas, D., Lo, A. W. (1998). Optimal control of execution costs. Journal of Financial Markets, 1, 1-50.

[8] Bertsimas, D., Hummel, P., Lo, A. W. (1999). Optimal control of execution costs for portfolios. Computing in science and engineering, 40-54.

[9] Chan, L. K. C., Lakonishok, J. (1993). Institutional trades and intraday stock price behavior. Journal of Financial Economics, 33, 173-199.

[10] Chan, L. K. C., Lakonishok, J (1995). The behavior of stock prices around institutional trades. The Journal of Finance, 50(4), 1147-1174.

[11] Darvas, Z., von Weizsäcker, J. (2011). Financial transaction tax: small is beautiful. Society and Economy, 33(3), 449-473.

[12] Goldstein, M. A., Irvine, P., Kandel, E., Wienern, Z. (2009). Brokerage commissions and institutional trading patterns. Review of Financial Studies, 22(12), 5175-5212.

[13] Harris, L. (2003). Trading and Exchanges: Market Microstructure for Practitioners. Oxford University Press.

[14] He, H., Mamaysky, H. (2005). Dynamic trading policies with price impact. Journal of Economic Dynamics and Control 29,, 891-930.

[15] Holthausen, R. W., Leftwich, R. W., Mayers, D. (1990). Large-block transactions, the speed of response, and temporary and permanent stock-price effects. Journal of Financial Economics, 26, 71-95.

[16] Hu, G. (2009). Measures of implicit trading costs and buy-sell asymmetry. Journal of Financial Markets, 12, 418-437.

[17] Huberman, G., Stanzel, W. (2004). Arbitrage-free price update and price-impact functions. Econometrica, 72(4), 1247-1275.

[18] Keynes, J. M., (1936). The General Theory of Employment, Interest and Money. MacMillan, London.

[19] Kissell, R., Glantz, M. (2003). Optimal trading strategies: Quantitative approaches for managing market impact and trading risk. New York: Amacom.

[20] Kraus, A., Stoll, H. R. (1972). Price Impacts of Block Trading on the New York Stock Exchange. The Journal of Finance, 27(3), 569-588.

[21] Lee, C., Mucklow, B., Ready, M. (1993). Spreads, Depths and the impact of earnings information: An intraday Analysis. The Review of Financial Studies, 6(2), 345-374.

[22] Madhavan, A. (2002). VWAP strategies. In B. Bruce (Ed.), Investment Guides, Transaction Performance: The Changing Face of Trading (pp. 32-38). New York: Institutional Investor Inc.

[23] McInish, T., Wood, R. (1992). An analysis of intraday patterns in bid/ask spreads for NYSE stocks. The Journal of Finance, 47(2), 753-764.

[24] Obizhaeva, A., Wang, J. (2012). Optimal trading strategy and supply/demand dynamics, Journal of Financial Market, forthcoming.

[25] Patell, J. M., Wolfson, M. A. (1984). The intraday speed of adjustment of stock prices to earnings and dividend announcements. Journal of Financial Economics, 13, 223-252.

[26] Plerou, V., Gopikrishnan, P., Stanley H. E. (2005). Quantifying fluctuations in market liquidity: Analysis of the bid-ask spread. Physical Review E, 71, 046131.

[27] Ponzi, A., Lillo, F., Mantegna R. N. (2009). Market reaction to a bid-ask spread change: A power-law relaxation dynamics. Physical Review E, 80, 016112.

[28] Schöneborn, T., Schied, A. (2009). Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. Finance and Stochastics, 13, 181-204.

[29] Schulmeister, S. (2009). A general financial transaction tax: a short cut of the pros, the cons and a proposal. WIFO Working Paper No. 344.

[30] Schulmeister, S., Schratzenstaller, M., Picek, O. (2008). A general financial transaction tax: motives, revenues, feasibility and effects. WIFO Working Paper.

[31] Schwartz, T., R. (1988). Equity Markets. Harper and Row: New York.

[32] Sun, E. W., Kruse, T., Yu, M. T. (2012). High frequency trading, liquidity, and execution cost. Working paper.

[33] Ting, C. (2006). Which daily price is less noisy? Financial Management, 35, 81-95.

[34] Ting, C., Warachka, M., Zhao, Y. (2007). Optimal liquidation strategies and their implications. Journal of Economic Dynamics and Control, 31, 1431-1450.

[35] Tobin, J. (1978). A proposal for international monetary reform. Eastern Economic Journal, 4(3-4), 153-159.

[36] Werner, I. M. (2003). NYSE order flow, spreads, and information. Journal of Financial Markets, 6, 309-335.

Appendix

A. Proof of Proposition 1

We use induction to prove the Proposition 1. For the optimal value function of the last time period we obtain

$$
\begin{aligned}
J_T(X_T, D_T, F_T, T, x_{t_{N-1}}) &= (F_T + \frac{S(x_{t_{N-1}})}{2})X_T + [\lambda(X_0 - X_T) + D_T + \frac{X_T}{2q}]X_T \\
&= (F_T + \frac{\alpha x_{n-1} + \beta}{2})X_T + [\lambda(X_0 - X_T) + D_T + \frac{X_T}{2q}]X_T.
\end{aligned}
$$

This is Equation 2 with Equation 4. We now verify that for every $t_n \in \{t_0, \ldots, t_N\}$ the optimal value function has the form given by Equation 2. We assume that the Proposition 1 holds true for some $t_{n+1} \in \{t_1, \ldots, t_N\}$ and then find for t_n,

$$
\begin{aligned}
J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) &= \overset{min}{x_{t_n}} \{[(F_{t_n} + \frac{S(x_{t_{n-1}})}{2}) + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{x_{t_n}}{2q}]x_{t_n} \\
&\quad + E_{t_n} J_{t_{n+1}}(X_{t_n} - x_{t_n}, (D_{t_n} + \kappa x_{t_n})e^{-\rho\tau}, F_{t_{n+1}}, t_{n+1}, x_{t_n})\} \\
&= \overset{min}{x_{t_n}} \{[(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2}) + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{x_{t_n}}{2q}]x_{t_n} \\
&\quad + (F_n + \frac{\beta}{2})(X_{t_n} - x_{t_n}) + \lambda X_0(X_{t_n} - x_{t_n}) + d_{n+1}(X_{t_n} - x_{t_n})^2 \\
&\quad + l_{n+1}(X_{t_n} - x_{t_n})(D_{t_n} + \kappa x_{t_n})e^{-\rho\tau} + f_{n+1}(D_{t_n} + \kappa x_{t_n})^2 e^{-2\rho\tau} \\
&\quad + h_{n+1}(X_{t_n} - x_{t_n})x_{t_n} + s_{n+1}(D_{t_n} + \kappa x_{t_n})e^{-\rho\tau} x_{t_n} + u_{n+1}x_{t_n}^2\}. \quad (9)
\end{aligned}
$$

To obtain the minimum we differentiate Equation 9 with respect to x_{t_n}

$$
\begin{aligned}
\frac{\partial J}{\partial x_{t_n}} &= F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2} + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{x_{t_n}}{q} \\
&\quad - (F_n + \frac{\beta}{2}) - \lambda X_0 - 2d_{n+1}(X_{t_n} - x_{t_n}) \\
&\quad + l_{n+1}(\kappa X_{t_n} - \kappa x_{t_n} - D_{t_n} - \kappa x_{t_n})e^{-\rho\tau} + f_{n+1}2\kappa(D_{t_n} + \kappa x_{t_n})e^{-2\rho\tau} \\
&\quad + h_{n+1}(X_{t_n} - 2x_{t_n}) + s_{n+1}(D_{t_n} + 2\kappa x_{t_n})e^{-\rho\tau} + 2u_{n+1}x_{t_n} \\
&= 2x_{t_n}[\frac{1}{2q} + d_{n+1} - \kappa l_{n+1}e^{-\rho\tau} + \kappa^2 f_{n+1}e^{-2\rho\tau} - h_{n+1} + \kappa s_{n+1}e^{-\rho\tau} + u_{n+1}] \\
&\quad + \frac{\alpha x_{t_{n-1}}}{2} \\
&\quad + X_{t_n}[-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1}] \\
&\quad + D_{t_n}[1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}].
\end{aligned}
$$

Seting $\frac{\partial J}{\partial x_{t_n}} \overset{!}{=} 0$ we obtain the optimal choice

$$
x_{t_n} = wX_{t_n} + vD_{t_n} + m, \quad (10)
$$

where

$$
\begin{aligned}
w &= -\frac{1}{2}\delta_{n+1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1}),\\
v &= -\frac{1}{2}\delta_{n+1}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}),\\
m &= -\frac{1}{2}\delta_{n+1}(\frac{\alpha x_{t_{n-1}}}{2}),\\
\delta_{n+1} &= (\frac{1}{2q} + d_{n+1} - \kappa l_{n+1}e^{-\rho\tau} + \kappa^2 f_{n+1}e^{-2\rho\tau} - h_{n+1} + \kappa s_{n+1}e^{-\rho\tau} + u_{n+1})^{-1}. \qquad (11)
\end{aligned}
$$

This proves Equation 1 of Proposition 1. Putting Equation 10 into Equation 9 we find for the optimal value function

$$
\begin{aligned}
&J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}})\\
&= [(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2}) + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{wX_{t_n} + vD_{t_n} + m}{2q}](wX_{t_n} + vD_{t_n} + m)\\
&+(F_n + \frac{\beta}{2})(X_{t_n} - wX_{t_n} - vD_{t_n} - m) + \lambda X_0(X_{t_n} - wX_{t_n} - vD_{t_n} - m)\\
&+d_{n+1}(X_{t_n} - wX_{t_n} - vD_{t_n} - m)^2 + l_{n+1}(X_{t_n} - wX_{t_n} - vD_{t_n} - m)(D_{t_n} + \kappa(wX_{t_n} + vD_{t_n} + m))e^{-\rho\tau}\\
&+f_{n+1}(D_{t_n} + \kappa(wX_{t_n} + vD_{t_n} + m))^2e^{-2\rho\tau} + h_{n+1}(X_{t_n} - wX_{t_n} - vD_{t_n} - m)(wX_{t_n} + vD_{t_n} + m)\\
&+s_{n+1}(D_{t_n} + \kappa(wX_{t_n} + vD_{t_n} + m))e^{-\rho\tau}(wX_{t_n} + vD_{t_n} + m) + u_{n+1}(wX_{t_n} + vD_{t_n} + m)^2\\
&= X_{t_n}^2(-\lambda w + \frac{w^2}{2q} + d_{n+1}(1-w)^2 + \kappa l_{n+1}e^{-\rho\tau}w(1-w) + \kappa^2 f_{n+1}e^{-2\rho\tau}w^2\\
&+h_{n+1}(1-w) + \kappa s_{n+1}e^{-\rho\tau}w^2 + u_{n+1}w^2)\\
&+D_{t_n}^2(v + \frac{v^2}{2q} + d_{n+1}v^2 - l_{n+1}e^{-\rho\tau}v(1+\kappa v) + f_{n+1}e^{-2\rho\tau}(1+\kappa v)^2 - h_{n+1}v^2\\
&+s_{n+1}e^{-\rho\tau}v(1+\kappa v) + u_{n+1}v^2)\\
&+X_{t_n}D_{t_n}(-\lambda v + w + \frac{wv}{2q} - 2d_{n+1}(1-w)v - l_{n+1}e^{-\rho\tau}((1-w)(1+\kappa v) - \kappa vw)\\
&+2\kappa f_{n+1}e^{-2\rho\tau}w(1+\kappa v) + h_{n+1}v(1-w) - h_{n+1}vw + \kappa s_{n+1}e^{-\rho\tau}vw\\
&+s_{n+1}e^{-\rho\tau}w(1+\kappa v) + 2u_{n+1}wv)\\
&+X_{t_n}(w(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2} + \lambda X_0) - \lambda m + \frac{mw}{q} + (1-w)(F_{t_n} + \frac{\beta}{2})\\
&+\lambda X_0(1-w) - 2d_{n+1}m(1-w) - l_{n+1}e^{-\rho\tau}(\kappa m(1-w) - \kappa wm)\\
&+2\kappa^2 f_{n+1}e^{-2\rho\tau}wm + h_{n+1}m(1-w) - h_{n+1}mw + 2\kappa s_{n+1}e^{-\rho\tau}mw + 2u_{n+1}mw)\\
&+D_{t_n}(v(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2} + \lambda X_0) + m + \frac{mw}{q} - v(F_{t_n} + \frac{\beta}{2}) - \lambda X_0 v\\
&+2d_{n+1}mv + l_{n+1}e^{-\rho\tau}(-(1+\kappa v)m - \kappa vm) + 2\kappa f_{n+1}e^{-2\rho\tau}(1+\kappa v)m\\
&-2h_{n+1}mv - h_{n+1}mw + s_{n+1}e^{-\rho\tau}m(1+\kappa v) + s_{n+1}e^{-\rho\tau}m\kappa v + 2u_{n+1}mv)\\
&+(m(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2} + \lambda X_0) + \frac{m^2}{2q} - m(F_{t_n} + \frac{\beta}{2}) - \lambda X_0 m + d_{n+1}m^2\\
&-l_{n+1}\kappa e^{-\rho\tau}m^2 + \kappa^2 m^2 f_{n+1}e^{-2\rho\tau} - h_{n+1}m^2 + s_{n+1}e^{-\rho\tau}m^2\kappa + u_{n+1}m^2).
\end{aligned}
$$

We then simplify the coefficients.

For $X_{t_n}^2$, we obtain:

$$\begin{aligned}
& -\lambda w + \frac{w^2}{2q} + d_{n+1}(1-w)^2 + \kappa l_{n+1}e^{-\rho\tau}w(1-w) + \kappa^2 f_{n+1}e^{-2\rho\tau}w^2 \\
& + h_{n+1}(1-w) + \kappa s_{n+1}e^{-\rho\tau}w^2 + u_{n+1}w^2 \\
= \; & d_{n+1} + w(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1}) \\
& + w^2(\frac{1}{2q} + d_{n+1} - \kappa l_{n+1}e^{-\rho\tau} + \kappa^2 f_{n+1}e^{-2\rho\tau} - h_{n+1} + \kappa s_{n+1}e^{-\rho\tau} + u_{n+1}) \\
= \; & d_{n+1} - \frac{1}{2}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})^2 \\
& + \frac{1}{4}\delta_{n+1}^{-2}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})^2\delta_{n+1} \\
= \; & d_{n+1} - \frac{1}{4}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})^2.
\end{aligned}$$

For $D_{t_n}^2$, we obtain:

$$\begin{aligned}
& v + \frac{v^2}{2q} + d_{n+1}v^2 - l_{n+1}e^{-\rho\tau}v(1+\kappa v) + f_{n+1}e^{-2\rho\tau}(1+\kappa v)^2 - h_{n+1}v^2 \\
& + s_{n+1}e^{-\rho\tau}v(1+\kappa v) + u_{n+1}v^2 \\
= \; & f_{n+1}e^{-2\rho\tau} + v(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}) \\
& + v^2(\frac{1}{2q} + d_{n+1} - \kappa l_{n+1}e^{-\rho\tau} + \kappa^2 f_{n+1}e^{-2\rho\tau} - h_{n+1} + \kappa s_{n+1}e^{-\rho\tau} + u_{n+1}) \\
= \; & f_{n+1}e^{-2\rho\tau} - \frac{1}{2}\delta_{n+1}^{-1}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau})^2 \\
& + \frac{1}{4}\delta_{n+1}^{-2}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau})^2\delta_{n+1} \\
= \; & f_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}^{-1}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau})^2.
\end{aligned}$$

For $X_{t_n}D_{t_n}$, we obtain:

$$\begin{aligned}
& -\lambda v + w + \frac{wv}{2q} - 2d_{n+1}(1-w)v - l_{n+1}e^{-\rho\tau}((1-w)(1+\kappa v) - \kappa vw) \\
& + 2\kappa f_{n+1}e^{-2\rho\tau}w(1+\kappa v) + h_{n+1}v(1-w) - h_{n+1}vw + \kappa s_{n+1}e^{-\rho\tau}vw \\
& + s_{n+1}e^{-\rho\tau}w(1+\kappa v) + 2u_{n+1}wv \\
= \; & l_{n+1}e^{-\rho\tau} + w(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}) \\
& + v(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1}) \\
& + 2vw(\frac{1}{2q} + d_{n+1} - \kappa l_{n+1}e^{-\rho\tau} + \kappa^2 f_{n+1}e^{-2\rho\tau} - h_{n+1} + \kappa s_{n+1}e^{-\rho\tau} + u_{n+1}) \\
= \; & l_{n+1}e^{-\rho\tau} \\
& - \frac{1}{2}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}) \\
& - \frac{1}{2}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}) \\
& + 2\frac{1}{4}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}) \\
= \; & l_{n+1}e^{-\rho\tau} - \frac{1}{2}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}).
\end{aligned}$$

For X_{t_n}, we obtain:

$$\begin{aligned}
& w(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2} + \lambda X_0) - \lambda m + \frac{mw}{q} + (1-w)(F_{t_n} + \frac{\beta}{2}) + \lambda X_0(1-w) \\
& -2d_{n+1}m(1-w) - l_{n+1}e^{-\rho\tau}(\kappa m(1-w) - \kappa w m) + 2\kappa^2 f_{n+1}e^{-2\rho\tau} wm \\
& +h_{n+1}m(1-w) - h_{n+1}mw + 2\kappa s_{n+1}e^{-\rho\tau}mw + 2u_{n+1}mw \\
= \;& (F_{t_n} + \frac{\beta}{2} + \lambda X_0) + w(\frac{\alpha x_{t_{n-1}}}{2}) + m(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1}) \\
& +2wm(\frac{1}{2q} + d_{n+1} - \kappa l_{n+1}e^{-\rho\tau} + \kappa^2 f_{n+1}e^{-2\rho\tau} - h_{n+1} + \kappa s_{n+1}e^{-\rho\tau} + u_{n+1}) \\
= \;& (F_{t_n} + \frac{\beta}{2} + \lambda X_0) - \frac{1}{2}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})(\frac{\alpha x_{t_{n-1}}}{2}) \\
& -\frac{1}{2}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})(\frac{\alpha x_{t_{n-1}}}{2}) \\
& +2\frac{1}{4}\delta_{n+1}^{-2}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})(\frac{\alpha x_{t_{n-1}}}{2})\delta_{n+1} \\
= \;& (F_{t_n} + \frac{\beta}{2} + \lambda X_0) - \frac{1}{2}\delta_{n+1}^{-1}(-\lambda - 2d_{n+1} + \kappa l_{n+1}e^{-\rho\tau} + h_{n+1})(\frac{\alpha x_{t_{n-1}}}{2}).
\end{aligned}$$

For D_{t_n}, we obtain:

$$\begin{aligned}
& v(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2} + \lambda X_0) + m + \frac{mw}{q} - v(F_{t_n} + \frac{\beta}{2}) - \lambda X_0 v + 2d_{n+1}mv \\
& +l_{n+1}e^{-\rho\tau}(-(1+\kappa v)m - \kappa v m) + 2\kappa f_{n+1}e^{-2\rho\tau}(1+\kappa v)m - 2h_{n+1}mv - h_{n+1}mw \\
& +s_{n+1}e^{-\rho\tau}m(1+\kappa v) + s_{n+1}e^{-\rho\tau}m\kappa v + 2u_{n+1}mv \\
= \;& v(\frac{\alpha x_{t_{n-1}}}{2}) + m(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau}) \\
& +2vm(\frac{1}{2q} + d_{n+1} - \kappa l_{n+1}e^{-\rho\tau} + \kappa^2 f_{n+1}e^{-2\rho\tau} - h_{n+1} + \kappa s_{n+1}e^{-\rho\tau} + u_{n+1}) \\
= \;& -\frac{1}{2}\delta_{n+1}^{-1}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau})(\frac{\alpha x_{t_{n-1}}}{2}) \\
& -\frac{1}{2}\delta_{n+1}^{-1}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau})(\frac{\alpha x_{t_{n-1}}}{2}) \\
& +2\frac{1}{4}\delta_{n+1}^{-2}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau})(\frac{\alpha x_{t_{n-1}}}{2})\delta_{n+1} \\
= \;& -\frac{1}{2}\delta_{n+1}^{-1}(1 - l_{n+1}e^{-\rho\tau} + 2\kappa f_{n+1}e^{-2\rho\tau} + s_{n+1}e^{-\rho\tau})(\frac{\alpha x_{t_{n-1}}}{2}).
\end{aligned}$$

Without X_{t_n} or D_{t_n}, we obtain:

$$\begin{aligned}
& m(F_{t_n} + \frac{\alpha x_{t_{n-1}} + \beta}{2} + \lambda X_0) + \frac{m^2}{2q} - m(F_{t_n} + \frac{\beta}{2}) - \lambda X_0 m + d_{n+1}m^2 \\
& -l_{n+1}\kappa e^{-\rho\tau}m^2 + \kappa^2 m^2 f_{n+1}e^{-2\rho\tau} - h_{n+1}m^2 + s_{n+1}e^{-\rho\tau}m^2\kappa + u_{n+1}m^2 \\
= \;& m(\frac{\alpha x_{t_{n-1}}}{2}) + m^2(\frac{1}{2q} + d_{n+1} - \kappa l_{n+1}e^{-\rho\tau} + \kappa^2 f_{n+1}e^{-2\rho\tau} - h_{n+1} + \kappa s_{n+1}e^{-\rho\tau} + u_{n+1}) \\
= \;& -\frac{1}{2}\delta_{n+1}^{-1}(\frac{\alpha x_{t_{n-1}}}{2})^2 + \frac{1}{4}\delta_{n+1}^{-2}(\frac{\alpha x_{t_{n-1}}}{2})^2\delta_{n+1} \\
= \;& -\frac{1}{4}\delta_{n+1}^{-1}(\frac{\alpha x_{t_{n-1}}}{2})^2.
\end{aligned}$$

From this we obtain Equation 2 with Equation 3. This concludes the proof.

B. Proof of Proposition 2

We use induction to prove the Proposition 2. For the last period we have

$$J_T(X_T, D_T, F_T, T, x_{T-1}) = (F_T + \frac{\alpha x_{T-1}+\beta}{2})X_T + [\lambda(X_0 - X_T) + D_T + \frac{X_T}{2q}]X_T.$$

This satisfies Equation 6 with Equation 8. Using that the Equation 6 holds true for some t_{n+1} we find for t_n,

$$\begin{aligned}
& J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) \\
= & \overset{min}{x_{t_n}} \{[(F_{t_n} + \frac{\alpha x_{n-1}+\beta}{2}) + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{x_{t_n}}{2q}]x_{t_n} \\
& + E_{t_n} J_{t_{n+1}}(X_{t_n} - x_{t_n}, (D_{t_n} + \kappa x_{t_n})e^{-\rho\tau}, F_{t_{n+1}}, t_{n+1}, x_{t_n})\} \\
= & \overset{min}{x_{t_n}} \{[(F_{t_n} + \frac{\alpha x_{n-1}+\beta}{2}) + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{x_{t_n}}{2q}]x_{t_n} \\
& + (a \times a^{N-(n+1)}F_n + \frac{\beta}{2})(X_{t_n} - x_{t_n}) + \lambda X_0(X_{t_n} - x_{t_n}) + b_{n+1}(X_{t_n} - x_{t_n})^2 \\
& + c_{n+1}(D_{t_n} + \kappa x_{t_n})^2 e^{-2\rho\tau} + d_{n+1} m F_{t_n}^2 + g_{n+1}(X_{t_n} - x_{t_n})(D_{t_n} + \kappa x_{t_n})e^{-\rho\tau} \\
& + h_{n+1}(X_{t_n} - x_{t_n})aF_{t_n} + l_{n+1}(D_{t_n} + \kappa x_{t_n})e^{-\rho\tau} aF_{t_n} \\
& + v_{n+1}(X_{t_n} - x_{t_n})x_{t_n} + w_{n+1} aF_{t_n} x_{t_n} + s_{n+1}(D_{t_n} + \kappa x_{t_n})e^{-\rho\tau} x_{t_n} + u_{n+1} x_{t_n}^2\}.
\end{aligned} \tag{12}$$

To obtain the minimum we differentiate the Equation 12 with respect to x_{t_n}

$$\begin{aligned}
\frac{\partial J}{\partial x_{t_n}} = & (F_{t_n} + \frac{\alpha x_{n-1}+\beta}{2}) + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{x_{t_n}}{q} \\
& -(a^{N-n}F_n + \frac{\beta}{2}) - \lambda X_0 - 2b_{n+1}(X_{t_n} - x_{t_n}) \\
& + 2\kappa c_{n+1}(D_{t_n} + \kappa x_{t_n})e^{-2\rho\tau} + g_{n+1}e^{-\rho\tau}[\kappa(X_{t_n} - x_{t_n}) - (D_{t_n} + \kappa x_{t_n})] - h_{n+1}aF_{t_n} \\
& + l_{n+1}\kappa e^{-\rho\tau} aF_{t_n} + v_{n+1}(X_{t_n} - 2x_{t_n}) + w_{n+1}aF_{t_n} + s_{n+1}(D_{t_n} + \kappa 2x_{t_n})e^{-\rho\tau} + 2u_{n+1}x_{t_n}\} \\
= & x_{t_n}(\frac{1}{q} + 2b_{n+1} - 2g_{n+1}\kappa e^{-\rho\tau} + c_{n+1}2\kappa^2 e^{-2\rho\tau} - 2v_{n+1} + 2s_{n+1}\kappa e^{-\rho\tau} + 2u_{n+1}) \\
& + X_{t_n}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa + v_{n+1}) + D_{t_n}(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}) \\
& + F_{t_n}(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a) + \frac{\alpha x_{t_{n-1}}}{2}.
\end{aligned}$$

Setting $\frac{\partial J}{\partial x_{t_n}} \overset{!}{=} 0$ we obtain the optimal choice

$$x_{t_n} = \gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi, \tag{13}$$

where

$$\begin{aligned}
\gamma &= -\frac{1}{2}\delta_{n+1}(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}), \\
\varphi &= -\frac{1}{2}\delta_{n+1}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa + v_{n+1}), \\
\eta &= -\frac{1}{2}\delta_{n+1}(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a), \\
\varpi &= -\frac{1}{2}\delta_{n+1}(\frac{\alpha x_{t_{n-1}}}{2}), \\
\delta_{n+1} &= (\frac{1}{2q} + b_{n+1} - g_{n+1}\kappa e^{-\rho\tau} + c_{n+1}\kappa^2 e^{-2\rho\tau} - v_{n+1} + s_{n+1}\kappa e^{-\rho\tau} + u_{n+1})^{-1}.
\end{aligned} \tag{14}$$

From this we find, that Equation 5 holds true. We then obtain

$$
\begin{aligned}
& J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) \\
= \; & [(F_{t_n} + \frac{\alpha x_{n-1} + \beta}{2}) + \lambda(X_0 - X_{t_n}) + D_{t_n} + \frac{(\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi)}{2q}](\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi) \\
& + (a \times a^{N-(n+1)} F_n + \frac{\beta}{2})(X_{t_n} - (\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi)) \\
& + \lambda X_0 (X_{t_n} - (\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi)) + b_{n+1}(X_{t_n} - (\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi))^2 \\
& + c_{n+1}(D_{t_n} + \kappa(\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi))^2 e^{-2\rho\tau} + d_{n+1} m F_{t_n}^2 \\
& + g_{n+1}(X_{t_n} - (\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi))(D_{t_n} + \kappa(\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi)) e^{-\rho\tau} \\
& + h_{n+1}(X_{t_n} - (\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi)) a F_{t_n} + l_{n+1}(D_{t_n} + \kappa(\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi)) e^{-\rho\tau} a F_{t_n} \\
& + v_{n+1}(X_{t_n} - (\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi))(\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi) \\
& + s_{n+1}(D_{t_n} + \kappa(\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi)) e^{-\rho\tau}(\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi) \\
& + u_{n+1}(\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi)^2 + w_{n+1} a F_{t_n}(\gamma D_{t_n} + \varphi X_{t_n} + \eta F_{t_n} + \varpi).
\end{aligned}
$$

And sort

$$
\begin{aligned}
& J_{t_n}(X_{t_n}, D_{t_n}, F_{t_n}, t_n, x_{t_{n-1}}) \\
= \; & D_{t_n}(\gamma\frac{\beta}{2} + \lambda X_0\gamma + \frac{2\gamma\varpi}{2q} + \varpi - \gamma\frac{\beta}{2} - \lambda X_0\gamma + 2b_{n+1}\gamma\varpi + 2c_{n+1}e^{-2\rho\tau}(1+\kappa\gamma)\kappa\varpi \\
& \quad -g_{n+1}e^{-\rho\tau}(\kappa\varpi\gamma + \varpi(1+\kappa\gamma)) - v_{n+1}2\gamma\varpi + s_{n+1}e^{-\rho\tau}(2\kappa\gamma\varpi + \varpi) + 2u_{n+1}\gamma\varpi) \\
& +F_{t_n}(\eta\frac{\beta}{2} + \lambda X_0\eta + \frac{2\eta\varpi}{2q} + \varpi - a^{N-n}\varpi - \eta\frac{\beta}{2} - \lambda X_0\eta + 2b_{n+1}\eta\varpi + 2c_{n+1}e^{-2\rho\tau}\kappa^2\eta\varpi \\
& \quad -2g_{n+1}e^{-\rho\tau}\kappa\eta\varpi - h_{n+1}a\varpi + l_{n+1}e^{-\rho\tau}a\kappa\varpi - v_{n+1}2\eta\varpi + 2s_{n+1}e^{-\rho\tau}\kappa\eta\varpi + 2u_{n+1}\eta\varpi + w_{n+1}a\varpi) \\
& +X_{t_n}(\varphi\frac{\beta}{2} + \lambda X_0\varphi + \frac{2\varphi\varpi}{2q} - \lambda\varpi + \frac{\beta}{2} - \varphi\frac{\beta}{2} + \lambda X_0 - \lambda X_0\varphi - 2b_{n+1}(1-\varphi)\varpi + 2c_{n+1}e^{-2\rho\tau}\kappa^2\varphi\varpi \\
& \quad +g_{n+1}e^{-\rho\tau}((1-\varphi)\kappa\varpi - \varpi\kappa\varphi) - v_{n+1}(2\varphi\varpi - \varpi) + 2s_{n+1}e^{-\rho\tau}\kappa\varphi\varpi + 2u_{n+1}\varphi\varpi) \\
& +D_{t_n}^2(\gamma + \frac{\gamma^2}{2q} + b_{n+1}\gamma^2 + c_{n+1}e^{-2\rho\tau}(1+\kappa\gamma)^2 - g_{n+1}e^{-\rho\tau}\gamma(1+\kappa\gamma) - v_{n+1}\gamma^2 \\
& \quad +s_{n+1}e^{-\rho\tau}(\kappa\gamma^2 + \gamma) + u_{n+1}\gamma^2) \\
& +F_{t_n}^2(\eta + \frac{\eta^2}{2q} - a^{N-n}\eta + b_{n+1}\eta^2 + c_{n+1}e^{-2\rho\tau}\kappa^2\eta^2 + d_{n+1}m - g_{n+1}e^{-\rho\tau}\kappa\eta^2 - h_{n+1}a\eta \\
& \quad +l_{n+1}e^{-\rho\tau}a\kappa\eta - v_{n+1}\eta^2 + s_{n+1}e^{-\rho\tau}\kappa\eta^2 + u_{n+1}\eta^2 + w_{n+1}a\eta) \\
& +X_{t_n}^2(-\lambda\varphi + \frac{\varphi^2}{2q} + b_{n+1}(1-\varphi)^2 + c_{n+1}e^{-2\rho\tau}\kappa^2\varphi^2 + g_{n+1}e^{-\rho\tau}(1-\varphi)\kappa\varphi - v_{n+1}(\varphi^2 - \varphi) \\
& \quad +s_{n+1}e^{-\rho\tau}\kappa\varphi^2 + u_{n+1}\varphi^2) \\
& +D_{t_n}F_{t_n}(\gamma + \frac{2\gamma\eta}{2q} + \eta - a^{N-n}\gamma + 2b_{n+1}\gamma\eta + 2c_{n+1}e^{-2\rho\tau}(1+\kappa\gamma)\kappa\eta - g_{n+1}e^{-\rho\tau}(\kappa\eta\gamma + \eta(1+\kappa\gamma)) \\
& \quad -h_{n+1}a\gamma + l_{n+1}e^{-\rho\tau}a(1+\kappa\gamma) - v_{n+1}2\gamma\eta + s_{n+1}e^{-\rho\tau}(2\kappa\gamma\eta + \eta) + 2u_{n+1}\gamma\eta + w_{n+1}a\gamma) \\
& +X_{t_n}D_{t_n}(-\lambda\gamma + \frac{2\gamma\varphi}{2q} + \varphi - 2b_{n+1}(1-\varphi)\gamma + 2c_{n+1}e^{-2\rho\tau}(1+\kappa\gamma)\kappa\varphi \\
& \quad +g_{n+1}e^{-\rho\tau}((1-\varphi)(1+\kappa\gamma) - \gamma\kappa\varphi) - v_{n+1}(2\gamma\varphi - \gamma) + s_{n+1}e^{-\rho\tau}(2\kappa\gamma\varphi + \varphi) + 2u_{n+1}\gamma\varphi) \\
& +X_{t_n}F_{t_n}(\varphi + \frac{2\varphi\eta}{2q} - \lambda\eta + a^{N-n} - a^{N-n}\varphi - 2b_{n+1}(1-\varphi)\eta + 2c_{n+1}e^{-2\rho\tau}\kappa^2\varphi\eta \\
& \quad +g_{n+1}e^{-\rho\tau}((1-\varphi)\kappa\eta - \eta\kappa\varphi) + h_{n+1}a(1-\varphi) + l_{n+1}e^{-\rho\tau}a\kappa\varphi - v_{n+1}(2\varphi\eta - \eta) \\
& \quad +2s_{n+1}e^{-\rho\tau}\kappa\varphi\eta + 2u_{n+1}\varphi\eta + w_{n+1}a\varphi) \\
& +X_{t_n}x_{t_{n-1}}(\varphi\frac{\alpha}{2}) + D_{t_n}x_{t_{n-1}}(\gamma\frac{\alpha}{2}) + F_{t_n}x_{t_{n-1}}(\eta\frac{\alpha}{2}) + x_{t_{n-1}}(\varpi\frac{\alpha}{2}) \\
& +(\varpi\frac{\beta}{2} + \lambda X_0\varpi + \frac{\varpi^2}{2q} - \varpi\frac{\beta}{2} - \lambda X_0\varpi + b_{n+1}\varpi^2 + c_{n+1}e^{-2\rho\tau}\kappa^2\varpi^2 - g_{n+1}e^{-\rho\tau}\kappa\varpi^2 \\
& \quad -v_{n+1}\varpi^2 + s_{n+1}e^{-\rho\tau}\kappa\varpi^2 + u_{n+1}\varpi^2).
\end{aligned}
$$

For $D_{t_n}^2$, we obtain:

$$
\begin{aligned}
& \gamma + \frac{\gamma^2}{2q} + b_{n+1}\gamma^2 + c_{n+1}e^{-2\rho\tau}(1+\kappa\gamma)^2 - g_{n+1}e^{-\rho\tau}\gamma(1+\kappa\gamma) - v_{n+1}\gamma^2 \\
& +s_{n+1}e^{-\rho\tau}(\kappa\gamma^2 + \gamma) + u_{n+1}\gamma^2 \\
= \; & \gamma^2(\frac{1}{2q} + b_{n+1} + \kappa^2 c_{n+1}e^{-2\rho\tau} - g_{n+1}\kappa e^{-\rho\tau} - v_{n+1} + s_{n+1}\kappa e^{-\rho\tau} + u_{n+1}) \\
& +\gamma(1 + 2\kappa c_{n+1}e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}) + c_{n+1}e^{-2\rho\tau} \\
= \; & c_{n+1}e^{-2\rho\tau} + \gamma^2\delta_{n+1}^{-1} + \gamma(1 + 2\kappa c_{n+1}e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}) \\
= \; & c_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}(1 + 2\kappa c_{n+1}e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau})^2.
\end{aligned}
$$

For $X_{t_n}^2$, we obtain:

$$\begin{aligned}
& -\lambda\varphi + \frac{\varphi^2}{2q} + b_{n+1}(1-\varphi)^2 + c_{n+1}e^{-2\rho\tau}\kappa^2\varphi^2 + g_{n+1}e^{-\rho\tau}(1-\varphi)\kappa\varphi - v_{n+1}(\varphi^2-\varphi) \\
& +s_{n+1}e^{-\rho\tau}\kappa\varphi^2 + u_{n+1}\varphi^2 \\
= & \; b_{n+1} + \varphi^2(\frac{1}{2q} + b_{n+1} + c_{n+1}e^{-2\rho\tau}\kappa^2 - g_{n+1}\kappa e^{-\rho\tau} - v_{n+1} + s_{n+1}e^{-\rho\tau}\kappa + u_{n+1}) \\
& +\varphi(-\lambda - 2b_{n+1} + g_{n+1}\kappa e^{-\rho\tau} + v_{n+1}) \\
= & \; b_{n+1} + \varphi^2\delta_{n+1}^{-1} + \varphi(-\lambda - 2b_{n+1} + g_{n+1}\kappa e^{-\rho\tau} + v_{n+1}) \\
= & \; b_{n+1} - \frac{1}{4}\delta_{n+1}(-\lambda - 2b_{n+1} + g_{n+1}\kappa e^{-\rho\tau} + v_{n+1})^2.
\end{aligned}$$

For $F_{t_n}^2$, we obtain:

$$\begin{aligned}
& \eta + \frac{\eta^2}{2q} - a^{N-n}\eta + b_{n+1}\eta^2 + c_{n+1}e^{-2\rho\tau}\kappa^2\eta^2 + d_{n+1}m - g_{n+1}e^{-\rho\tau}\kappa\eta^2 - h_{n+1}a\eta \\
& +l_{n+1}e^{-\rho\tau}a\kappa\eta - v_{n+1}\eta^2 + s_{n+1}e^{-\rho\tau}\kappa\eta^2 + u_{n+1}\eta^2 + w_{n+1}a\eta \\
= & \; d_{n+1}m + \eta^2(\frac{1}{2q} + b_{n+1} + c_{n+1}e^{-2\rho\tau}\kappa^2 - g_{n+1}e^{-\rho\tau}\kappa - v_{n+1} + s_{n+1}e^{-\rho\tau}\kappa + u_{n+1}) \\
& +\eta(1 - a^{N-n} - h_{n+1}a + l_{n+1}e^{-\rho\tau}a\kappa + w_{n+1}a) \\
= & \; d_{n+1}m + \eta^2\delta_{n+1}^{-1} + \eta(1 - a^{N-n} - h_{n+1}a + l_{n+1}e^{-\rho\tau}a\kappa + w_{n+1}a) \\
= & \; d_{n+1}m - \frac{1}{4}\delta_{n+1}(1 - a^{N-n} - h_{n+1}a + l_{n+1}e^{-\rho\tau}a\kappa + w_{n+1}a)^2.
\end{aligned}$$

For $F_{t_n}D_{t_n}$, we obtain:

$$\begin{aligned}
& \gamma + \frac{2\gamma\eta}{2q} + \eta - a^{N-n}\gamma + 2b_{n+1}\gamma\eta + 2c_{n+1}e^{-2\rho\tau}(1+\kappa\gamma)\kappa\eta - g_{n+1}e^{-\rho\tau}(\kappa\eta\gamma + \eta(1+\kappa\gamma)) \\
& -h_{n+1}a\gamma + l_{n+1}e^{-\rho\tau}a(1+\kappa\gamma) - v_{n+1}2\gamma\eta + s_{n+1}e^{-\rho\tau}(2\kappa\gamma\eta + \eta) + 2u_{n+1}\gamma\eta + w_{n+1}a\gamma \\
= & \; \gamma(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a) + \eta(1 + 2c_{n+1}\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}) \\
& +\gamma\eta 2(\frac{1}{2q} + b_{n+1} + c_{n+1}\kappa^2 e^{-2\rho\tau} - g_{n+1}\kappa e^{-\rho\tau} - v_{n+1} + s_{n+1}\kappa e^{-\rho\tau} + u_{n+1}) + l_{n+1}e^{-\rho\tau}a \\
= & \; l_{n+1}e^{-\rho\tau}a \\
& -\frac{1}{2}\delta_{n+1}(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a)(1 + 2c_{n+1}\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}).
\end{aligned}$$

For $F_{t_n}X_{t_n}$, we obtain:

$$\begin{aligned}
& \varphi + \frac{2\varphi\eta}{2q} - \lambda\eta + a^{N-n} - a^{N-n}\varphi - 2b_{n+1}(1-\varphi)\eta + 2c_{n+1}e^{-2\rho\tau}\kappa^2\varphi\eta \\
& +g_{n+1}e^{-\rho\tau}((1-\varphi)\kappa\eta - \eta\kappa\varphi) + h_{n+1}a(1-\varphi) + l_{n+1}e^{-\rho\tau}a\kappa\varphi - v_{n+1}(2\varphi\eta - \eta) \\
& +2s_{n+1}e^{-\rho\tau}\kappa\varphi\eta + 2u_{n+1}\varphi\eta + w_{n+1}a\varphi \\
= & \; a^{N-n} + h_{n+1}a \\
& -\frac{1}{2}\delta_{n+1}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa + v_{n+1})(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a).
\end{aligned}$$

For $D_{t_n}X_{t_n}$, we obtain:

$$
\begin{aligned}
& -\lambda\gamma + \frac{2\gamma\varphi}{2q} + \varphi - 2b_{n+1}(1-\varphi)\gamma + 2c_{n+1}e^{-2\rho\tau}(1+\kappa\gamma)\kappa\varphi \\
& +g_{n+1}e^{-\rho\tau}((1-\varphi)(1+\kappa\gamma)-\gamma\kappa\varphi) - v_{n+1}(2\gamma\varphi-\gamma) + s_{n+1}e^{-\rho\tau}(2\kappa\gamma\varphi+\varphi) + 2u_{n+1}\gamma\varphi \\
= \ & g_{n+1}e^{-\rho\tau} \\
& -\frac{1}{2}\delta_{n+1}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa + v_{n+1})(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}).
\end{aligned}
$$

For D_{t_n} and $D_{t_n}x_{t_{n-1}}$, we obtain:

$$
\begin{aligned}
& x_{t_{n-1}}(\gamma\frac{\alpha}{2}) + \gamma\frac{\beta}{2} + \lambda X_0\gamma + \frac{2\gamma\varpi}{2q} + \varpi - \gamma\frac{\beta}{2} - \lambda X_0\gamma + 2b_{n+1}\gamma\varpi + 2c_{n+1}e^{-2\rho\tau}(1+\kappa\gamma)\kappa\varpi \\
& -g_{n+1}e^{-\rho\tau}(\kappa\varpi\gamma + \varpi(1+\kappa\gamma)) - v_{n+1}2\gamma\varpi + s_{n+1}e^{-\rho\tau}(2\kappa\gamma\varpi+\varpi) + 2u_{n+1}\gamma\varpi \\
= \ & -\frac{1}{2}\delta_{n+1}\frac{\alpha x_{t_{n-1}}}{2}(1 + c_{n+1}2\kappa e^{-2\rho\tau} - g_{n+1}e^{-\rho\tau} + s_{n+1}e^{-\rho\tau}).
\end{aligned}
$$

For X_{t_n} and $X_{t_n}x_{t_{n-1}}$, we obtain:

$$
\begin{aligned}
& x_{t_{n-1}}(\varphi\frac{\alpha}{2}) + \varphi\frac{\beta}{2} + \lambda X_0\varphi + \frac{2\varphi\varpi}{2q} - \lambda\varpi + \frac{\beta}{2} - \varphi\frac{\beta}{2} + \lambda X_0 - \lambda X_0\varphi - 2b_{n+1}(1-\varphi)\varpi + 2c_{n+1}e^{-2\rho\tau}\kappa^2\varphi\varpi \\
& +g_{n+1}e^{-\rho\tau}((1-\varphi)\kappa\varpi - \varpi\kappa\varphi) - v_{n+1}(2\varphi\varpi - \varpi) + 2s_{n+1}e^{-\rho\tau}\kappa\varphi\varpi + 2u_{n+1}\varphi\varpi \\
= \ & \lambda X_0 + \frac{\beta}{2} - \frac{1}{2}\delta_{n+1}\frac{\alpha x_{t_{n-1}}}{2}(-\lambda - 2b_{n+1} + g_{n+1}e^{-\rho\tau}\kappa + v_{n+1}).
\end{aligned}
$$

For F_{t_n} and $F_{t_n}x_{t_{n-1}}$, we obtain:

$$
\begin{aligned}
& x_{t_{n-1}}(\eta\frac{\alpha}{2}) + \eta\frac{\beta}{2} + \lambda X_0\eta + \frac{2\eta\varpi}{2q} + \varpi - a^{N-n}\varpi - \eta\frac{\beta}{2} - \lambda X_0\eta + 2b_{n+1}\eta\varpi + 2c_{n+1}e^{-2\rho\tau}\kappa^2\eta\varpi \\
& -2g_{n+1}e^{-\rho\tau}\kappa\eta\varpi - h_{n+1}a\varpi + l_{n+1}e^{-\rho\tau}a\kappa\varpi - v_{n+1}2\eta\varpi + 2s_{n+1}e^{-\rho\tau}\kappa\eta\varpi + 2u_{n+1}\eta\varpi + w_{n+1}a\varpi \\
= \ & -\frac{1}{2}\delta_{n+1}\frac{\alpha x_{t_{n-1}}}{2}(1 - a^{N-n} - h_{n+1}a + l_{n+1}\kappa e^{-\rho\tau}a + w_{n+1}a).
\end{aligned}
$$

For the remainder, we obtain:

$$
\begin{aligned}
& (\varpi\frac{\beta}{2} + \lambda X_0\varpi + \frac{\varpi^2}{2q} - \varpi\frac{\beta}{2} - \lambda X_0\varpi + b_{n+1}\varpi^2 + c_{n+1}e^{-2\rho\tau}\kappa^2\varpi^2 - g_{n+1}e^{-\rho\tau}\kappa\varpi^2 \\
& -v_{n+1}\varpi^2 + s_{n+1}e^{-\rho\tau}\kappa\varpi^2 + u_{n+1}\varpi^2) + x_{t_{n-1}}(\varpi\frac{\alpha}{2}) \\
= \ & \frac{1}{4}\delta_{n+1}(\frac{\alpha^2 x_{t_{n-1}}^2}{4}) - \frac{1}{2}\delta_{n+1}(\frac{\alpha^2 x_{t_{n-1}}^2}{4}) \\
= \ & -\frac{1}{4}\delta_{n+1}(\frac{\alpha^2 x_{t_{n-1}}^2}{4}).
\end{aligned}
$$

This proves Equation 6 and concludes the proof.

Table 1: Optimal order behavior of S-K-Y strategy for different values of α.

Underlying value	α	0	1	2	3	4	5	6	7	8	9	10
BM	α=0.01 bp	26799	4651	5490	5215	5231	5227	5231	5215	5490	4651	26799
	α=0.05 bp	28492	1562	7261	4686	5452	5094	5452	4686	7261	1562	28492
	α=0.1 bp	29313	0	9435	1996	7938	2639	7938	1996	9435	0	29313
	α=1 bp	29207	0	10397	0	10397	0	10397	0	10397	0	29207
GBM (μ=3%)	α=0.01 bp	41944	5639	6663	5889	5589	5255	4928	4592	4362	3710	11430
	α=0.05 bp	44559	706	9513	5049	5919	5122	5037	4380	5032	2496	12185
	α=0.1 bp	44946	0	11360	2122	8556	2656	7395	1895	7590	0	13479
	α=1 bp	44834	0	12391	0	11100	0	9801	0	8495	0	13379

Table 2: Optimal order behavior of S-K-Y strategy for different values of β

Underlying value	β	0	1	2	3	4	5	6	7	8	9	10
BM	β=10 bps	28492	1562	7261	4686	5452	5094	5452	4686	7261	1562	28492
	β=20 bps	28492	1562	7261	4686	5452	5094	5452	4686	7261	1562	28492
	β=30 bps	28492	1562	7261	4686	5452	5094	5452	4686	7261	1562	28492
GBM	β=10 bps	44559	706	9513	5049	5919	5122	5037	4380	5032	2496	12185
	β=20 bps	44559	706	9513	5049	5919	5122	5037	4380	5032	2496	12185
	β=30 bps	44559	706	9513	5049	5919	5122	5037	4380	5032	2496	12185

Table 3: Order sizes given by alternative strategy for all time points of trading for the case without a spread.

Trading Period	0	1	2	3	4	5	6	7	8	9	10
O-W	26474	5228	5228	5228	5228	5228	5228	5228	5228	5228	26474
Naive	9091	9091	9091	9091	9091	9091	9091	9091	9091	9091	9090

Table 4: Comparison of strategies using the expected value of VWAP and its variance (shown in parenthesis) for 10,000 runs for different combinations of parameters.

		μ=-10%	μ=-5%	μ=-1%	μ=1%	μ=5%	μ=10%
α=1 bp	S-K-Y (GBM)	99.7741 (2.6639)	104.1473 (1.9290)	106.8739 (0.9828)	107.8768 (0.6478)	109.1861 (0.2011)	110.0000 (0.0000)
	S-K-Y (BM)	102.3679 (0.8491)	104.8877 (0.8477)	106.9011 (0.8063)	107.9085 (0.8024)	109.9052 (0.8012)	112.3783 (0.7856)
	O-W	102.5928 (0.8368)	105.1166 (0.8299)	107.1418 (0.8217)	108.1249 (0.8209)	110.1165 (0.8084)	112.5933 (0.7791)
	Naive	102.9258 (1.0437)	105.5043 (0.9819)	107.5042 (0.9603)	108.5033 (0.9349)	110.5128 (0.9153)	112.9395 (0.8849)
α=5 bps	S-K-Y (GBM)	99.7604 (2.6021)	104.1734 (1.9429)	106.8521 (1.0077)	107.8748 (0.6485)	109.1877 (0.2015)	110.0000 (0.0000)
	S-K-Y (BM)	102.3555 (0.8363)	104.8880 (0.8335)	106.9092 (0.8061)	107.9128 (0.7932)	109.9029 (0.8060)	112.3664 (0.7750)
	O-W	103.5841 (0.8589)	106.1171 (0.8407)	108.1388 (0.8218)	109.1127 (0.8042)	111.1106 (0.8122)	113.5741 (0.7873)
	Naive	104.5834 (1.0064)	107.1402 (0.9935)	109.1755 (0.9622)	110.1537 (0.9473)	112.1535 (0.9268)	114.6037 (0.8966)
α=10 bps	S-K-Y (GBM)	99.8209 (2.6709)	104.1731 (1.9328)	106.8805 (1.0043)	107.8823 (0.6674)	109.1856 (0.2036)	110.0000 (0.0000)
	S-K-Y (BM)	102.3519 (0.8459)	104.8887 (0.8379)	106.8886 (0.8179)	107.8988 (0.8040)	109.8854 (0.7901)	112.3556 (0.7632)
	O-W	104.8232 (0.8632)	107.3621 (0.8476)	109.3607 (0.8232)	110.3600 (0.8177)	112.3476 (0.7973)	114.8071 (0.7743)
	Naive	106.6466 (1.0298)	109.1937 (0.9872)	111.2138 (0.9564)	112.2289 (0.9656)	114.2151 (0.9423)	116.6552 (0.8874)
α=50 bps	S-K-Y (GBM)	99.7712 (2.6490)	104.1742 (1.9554)	106.8787 (1.0122)	107.8755 (0.6556)	109.1772 (0.1963)	110.0000 (0.0000)
	S-K-Y (BM)	102.3579 (0.8464)	104.8887 (0.8255)	106.9053 (0.8273)	107.8981 (0.8212)	109.8908 (0.7971)	112.3600 (0.7965)
	O-W	114.7158 (0.8581)	117.2469 (0.8452)	119.2690 (0.8054)	120.2804 (0.8076)	122.2718 (0.7922)	124.7328 (0.7847)
	Naive	123.2002 (1.0029)	125.7389 (0.9893)	127.7583 (0.9583)	128.7656 (0.9557)	130.7514 (0.9309)	133.1721 (0.8752)
α=100 bps	S-K-Y (GBM)	99.7929 (2.6759)	104.1709 (1.9275)	106.8716 (1.0057)	107.8699 (0.6498)	109.1825 (0.1981)	110.0000 (0.0000)
	S-K-Y (BM)	102.3634 (0.8356)	104.8942 (0.8496)	106.9058 (0.8077)	107.9003 (0.8150)	109.8945 (0.7906)	112.3615 (0.7866)
	O-W	127.1105 (0.8586)	129.6532 (0.8486)	131.6746 (0.8280)	132.6599 (0.7909)	134.6438 (0.8084)	137.1115 (0.7728)
	Naive	143.8224 (1.0245)	146.3942 (1.0168)	148.4361 (0.9841)	149.4153 (0.9381)	151.4117 (0.9150)	153.8454 (0.8921)

Table 5: Comparison of strategies using the expected value of VWAP and its variance (shown in parenthesis) for 100,000 runs for different combinations of parameters.

		μ=-10%	μ=-5%	μ=-1%	μ=1%	μ=5%	μ=10%
α=1 bp	S-K-Y (GBM)	99.7774 (2.6571)	104.1736 (1.9258)	106.8686 (0.9994)	107.8751 (0.6494)	109.1802 (0.2001)	110.0000 (0.0000)
	S-K-Y (BM)	102.3601 (0.8473)	104.8879 (0.8344)	106.9025 (0.8165)	107.8980 (0.8046)	109.8903 (0.8016)	112.3609 (0.7882)
	O-W	102.5916 (0.8573)	105.1291 (0.8345)	107.1316 (0.8225)	108.1312 (0.8169)	110.1252 (0.8002)	112.5916 (0.7882)
	Naive	102.9343 (1.0281)	105.4872 (0.9847)	107.5122 (0.9586)	108.5072 (0.9426)	110.4945 (0.9202)	112.9388 (0.8948)
α=5 bps	S-K-Y (GBM)	99.7748 (2.6678)	104.1903 (1.9329)	106.8687 (1.0030)	107.8673 (0.6492)	109.1801 (0.2002)	110.0000 (0.0000)
	S-K-Y (BM)	102.3587 (0.8475)	104.8919 (0.8280)	106.9022 (0.8215)	107.8945 (0.8047)	109.8935 (0.7990)	112.3589 (0.7838)
	O-W	103.5756 (0.8526)	106.1139 (0.8300)	108.1277 (0.8140)	109.1259 (0.8131)	111.1131 (0.8087)	113.5788 (0.7916)
	Naive	104.5922 (1.0273)	107.1500 (0.9931)	109.1635 (0.9624)	110.1689 (0.9436)	112.1487 (0.9162)	114.5955 (0.8940)
α=10 bps	S-K-Y (GBM)	99.7866 (2.6541)	104.1836 (1.9338)	106.8678 (0.9976)	107.8763 (0.6472)	109.1833 (0.2006)	110.0000 (0.0000)
	S-K-Y (BM)	102.3602 (0.8466)	104.8905 (0.8323)	106.9028 (0.8201)	107.8992 (0.8115)	109.8938 (0.7938)	112.3624 (0.7864)
	O-W	104.8169 (0.8554)	107.3532 (0.8355)	109.3630 (0.8213)	110.3608 (0.8118)	112.3509 (0.8042)	114.8239 (0.7780)
	Naive	106.6542 (1.0354)	109.2137 (0.9829)	111.2318 (0.9565)	112.2349 (0.9525)	114.2108 (0.9148)	116.6577 (0.8966)
α=50 bps	S-K-Y (GBM)	99.7818 (2.6670)	104.1823 (1.9447)	106.8701 (1.0069)	107.8731 (0.6509)	109.1796 (0.1986)	110.0000 (0.0000)
	S-K-Y (BM)	102.3567 (0.8494)	104.8883 (0.8249)	106.9007 (0.8236)	107.9024 (0.8124)	109.8827 (0.7982)	112.3563 (0.7884)
	O-W	114.7292 (0.8551)	117.2671 (0.8320)	119.2722 (0.8209)	120.2737 (0.8146)	122.2672 (0.8078)	124.7355 (0.7881)
	Naive	123.1803 (1.0261)	125.7430 (0.9916)	127.7637 (0.9643)	128.7612 (0.9586)	130.7396 (0.9177)	133.1868 (0.9001)
α=100 bps	S-K-Y (GBM)	99.7850 (2.6478)	104.1815 (1.9257)	106.8699 (0.9974)	107.8727 (0.6482)	109.1805 (0.1992)	110.0000 (0.0000)
	S-K-Y (BM)	102.3555 (0.8484)	104.8824 (0.8371)	106.9025 (0.8194)	107.8943 (0.8092)	109.8918 (0.7988)	112.3644 (0.7848)
	O-W	127.1100 (0.8607)	129.6452 (0.8387)	131.6549 (0.8168)	132.6528 (0.8135)	134.6458 (0.8032)	137.1211 (0.7880)
	Naive	143.8443 (1.0235)	146.4020 (0.9855)	148.4170 (0.9630)	149.4183 (0.9385)	151.4015 (0.9251)	153.8470 (0.8938)

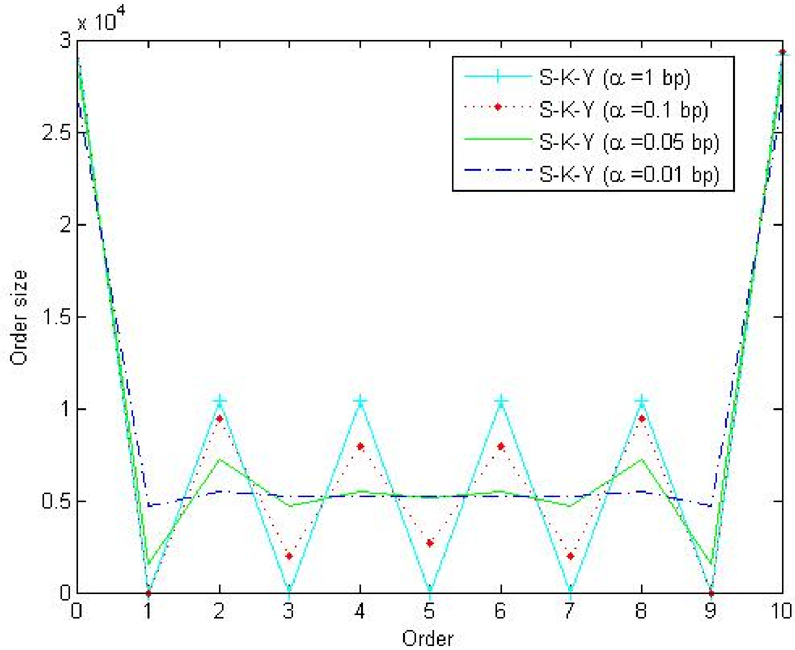

Figure 1: Optimal order behavior given by S-K-Y strategy for different values of α when underlying price dynamics following the Brownian motion.

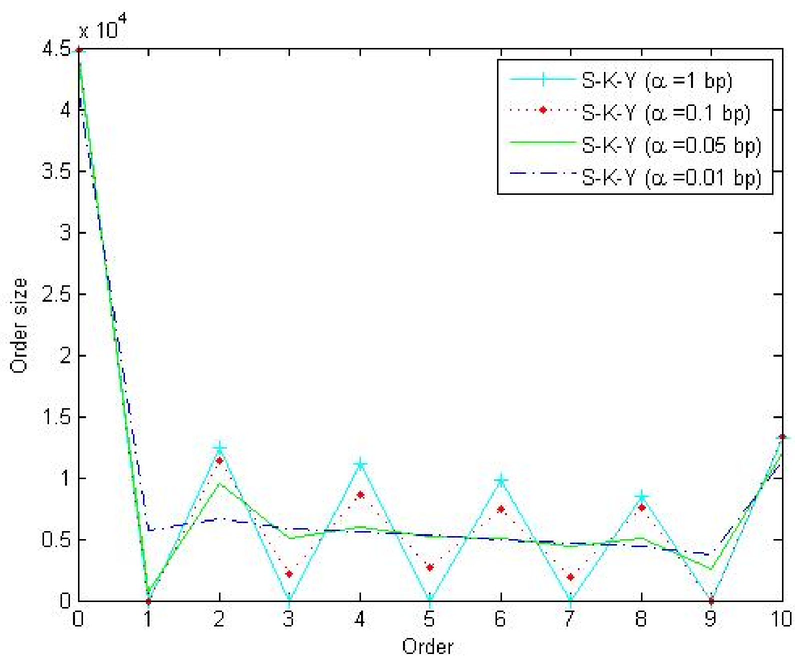

Figure 2: Optimal order behavior given by S-K-Y strategy for different values of α when underlying price dynamics following the geometric Brownian motion.

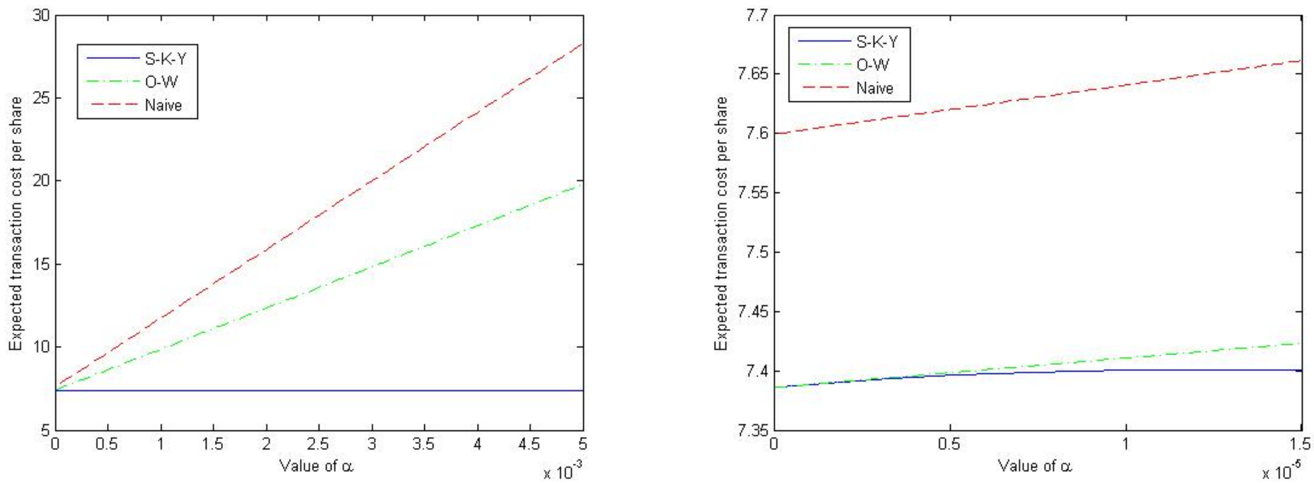

Figure 3: Expected transaction cost for diffent trading strategies when underlying price dynamics following the Brownian motion.

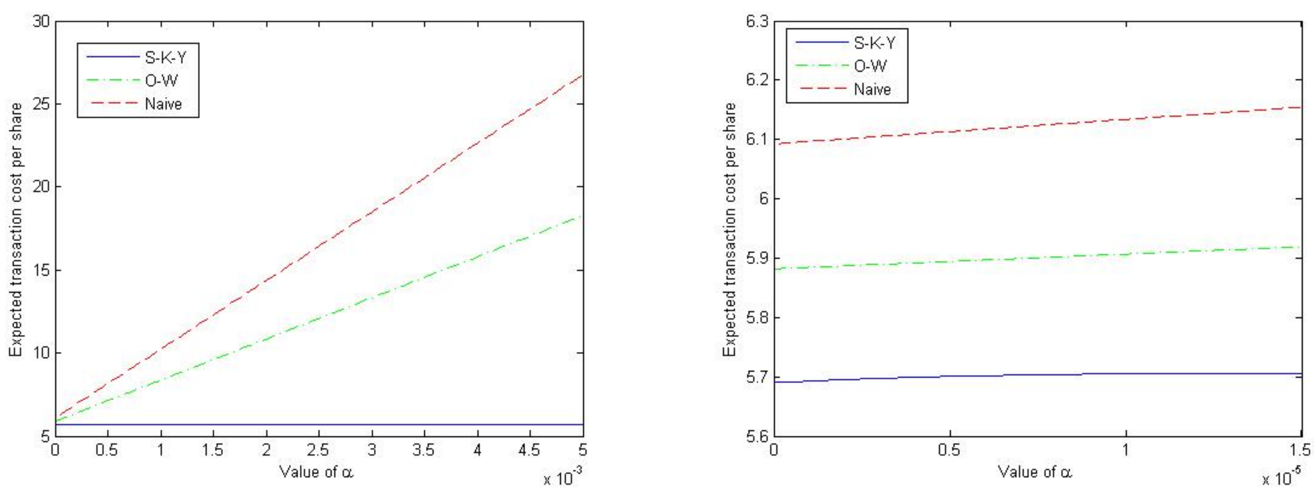

Figure 4: Expected transaction cost for diffent trading strategies when underlying price dynamics following the geometric Brownian motion ($\mu = -3\%$).

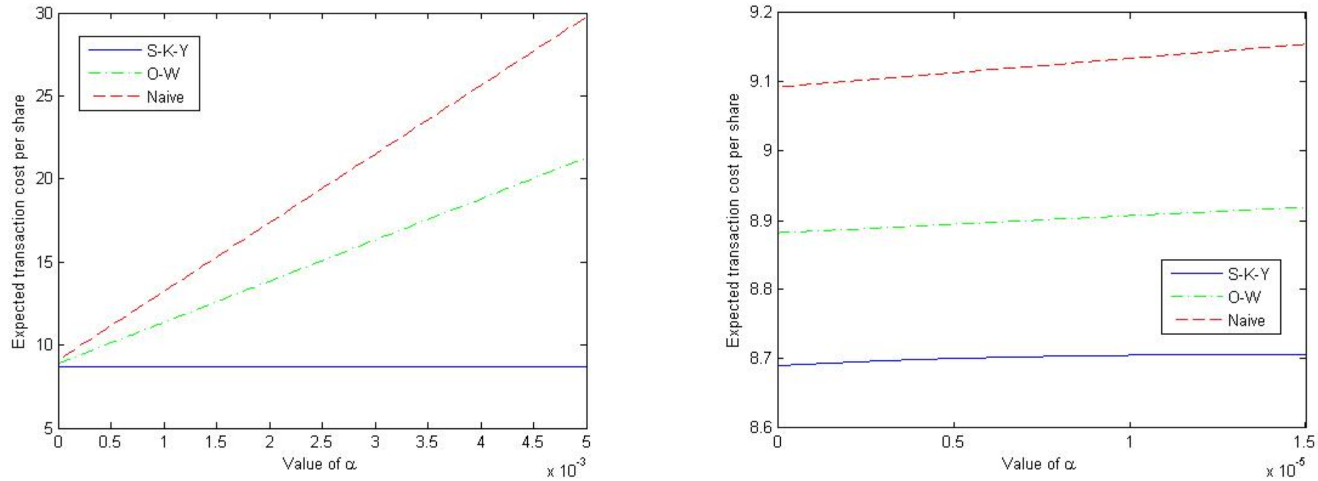

Figure 5: Expected transaction cost for diffent trading strategies when underlying price dynamics following the geometric Brownian motion ($\mu = 3\%$)

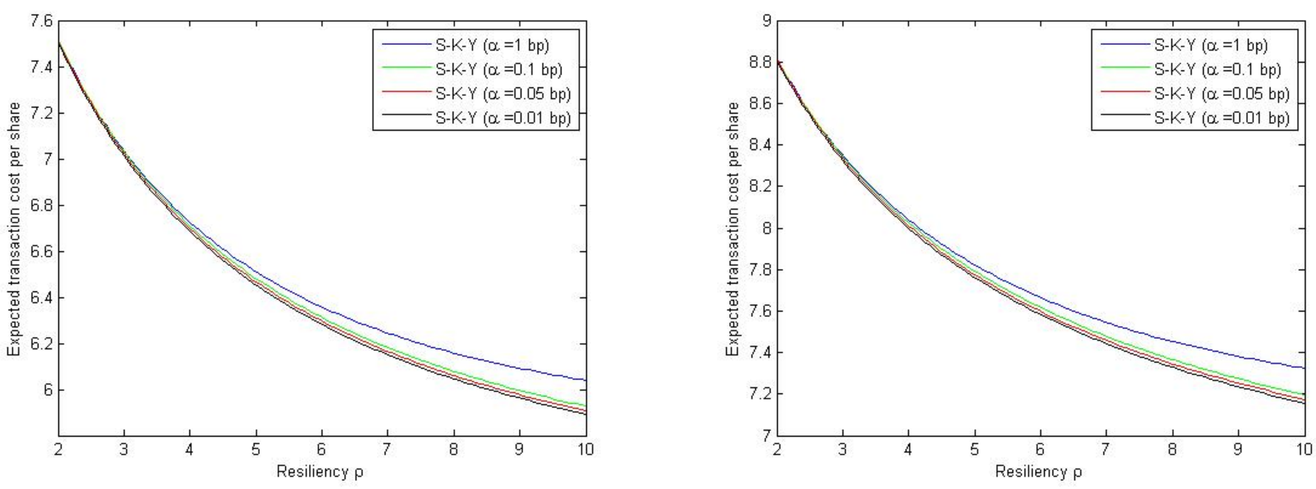

Figure 6: Expected transaction cost and liquidity measure ρ for different α. The left panel illustrates the case when the underlying price dynamics is a Brownian motion and the right panel for the geometric Brownian motion ($\mu = 3\%$).